1. Start with a sequence $\{a_n\}_{n=1}^{\infty}$. (In above Ex., $a_n = \frac{1}{2^n}$). Think of $\{a_n\}_n$ as an ordered list of numbers, i.e.,

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$$

2. Form the corresponding (formal) series $\sum_{n=1}^{\infty} a_n$. Think of the series $\sum a_n$ as

$$\sum_{n=1}^{\infty} a_n \ = \ a_1 + a_2 + a_3 + \ldots \ \stackrel{\text{note}}{=} \ \sum_{k=1}^{\infty} a_k \ .$$

3. Look at the corresponding n^{th} partial sum s_n where

$$s_n := a_1 + a_2 + \cdots + a_n \ \stackrel{\text{def}}{=} \ \sum_{k=1}^{n} a_k \quad \cancel{\stackrel{\text{NOT SO}}{=} \ \sum_{n=1}^{n} a_n} \ .$$

So we get the sequence of partial sums

$$\{s_n\}_{n=1}^{\infty} \ = \ \{ \underbrace{a_1}_{s_1}, \underbrace{a_1 + a_2}_{s_2}, \underbrace{a_1 + a_2 + a_3}_{s_3}, \underbrace{a_1 + a_2 + a_3 + a_4}_{s_4}, \underbrace{a_1 + a_2 + a_3 + a_4 + a_5}_{s_5}, \ \ldots \} \ .$$

4. Beware: for a series $\sum a_n$

- the n^{th} partial sum of $\sum a_n$ is $s_n \stackrel{\text{def}}{=} a_1 + a_2 + \ldots + a_n$
- the n^{th} term of $\sum a_n$ is a_n .

Since $s_n = (a_1 + \ldots + a_{n-1}) + a_n = s_{n-1} + a_n$, we have that relation that $a_n = s_n - s_{n-1}$.

5. We say that the infinite

5.1) series $\sum a_n$ converges provided the sequence of partial sums $\{s_n\}_n$ converges
5.2) series $\sum a_n$ diverges to $^+\infty$ provided the sequence of partial sums $\{s_n\}_n$ diverges to $^+\infty$
5.3) series $\sum a_n$ diverges to $^-\infty$ provided the sequence of partial sums $\{s_n\}_n$ diverges to $^-\infty$.
5.4) series $\sum a_n$ diverges provided the sequence of partial sums $\{s_n\}_n$ diverges

We write (in the first 3 cases, i.e., in 5.1, 5.2, and 5.3) as:

$$\sum_{n=1}^{\infty} a_n \ = \ \lim_{n\to\infty} s_n \ = \ \lim_{n\to\infty} (a_1 + a_2 + \ldots + a_n) \ = \ \lim_{n\to\infty} \sum_{k=1}^{n} a_k \ .$$

6. It doesn't matter where you start Theorem Note: $\sum_{k=1}^{N} a_k = (a_1 + a_2 + \ldots + a_{16}) + \sum_{k=17}^{N} a_k$.

So $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=17}^{\infty} a_n$ *do the same thing* amongst the choices from 5.1–5.4.

∘ $\sum_{n=1}^{\infty} a_n$ converges (to some finite number) $\Leftrightarrow$ $\sum_{n=17}^{\infty} a_n$ converges (to some finite number).
(warning: each series converges but the finite number they converge to may be different).

∘ $\sum_{n=1}^{\infty} a_n$ diverges to ∞ $\Leftrightarrow$ $\sum_{n=17}^{\infty} a_n$ diverges to ∞.

∘ $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$ $\Leftrightarrow$ $\sum_{n=17}^{\infty} a_n$ diverges to $-\infty$.

∘ $\sum_{n=1}^{\infty} a_n$ diverges $\Leftrightarrow$ $\sum_{n=17}^{\infty} a_n$ diverges.

7. n^{th}-term test for divergence.

Since: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
get the Test: If $\lim_{n\to\infty} a_n \neq 0$ (which includes the possiblity that $\lim_{n\to\infty} a_n$ DNE), then $\sum a_n$ diverges.
Warning: If $\lim_{n\to\infty} a_n = 0$, then it is possible that $\sum a_n$ converges and it is possible that $\sum a_n$ diverges.
Remark: The n^{th}-term test (for divergence) can show divergence but can NOT show convergence.

8. Let $\sum a_n$ is a positive termed series (which just means that each term $a_n \geq 0$). Then $s_n \leq s_{n+1}$ and so the the sequence $\{s_n\}_n$ is $\nearrow$, i.e., is nondecreasing and so

- EITHER $\{s_n\}_n$ converges (to some finite number), in which case $\sum a_n$ converges and $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$
- OR $\{s_n\}_n$ diverges to ∞, in which case $\sum a_n$ diverges to ∞ and $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \infty$