Theorem 1. *Power series vs. Taylor series.*
If f has a power series representation (expansion) about x_0, that is, if for some $R > 0$

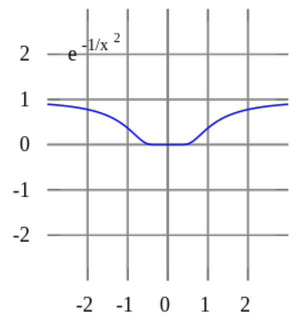
$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad \text{valid when} \quad |x - x_0| < R$$

then the c_n's must satisfy

$$c_n = \frac{f^{(n)}(x_0)}{n!} .$$

In short, if a function $y = f(x)$ has a power series representation about x_0,
then that power series representation must be the Taylor series $y = P_\infty(x)$ of f about x_0.

Warning. It is possible for a function not to be equal to it's Taylor series. Consider the function



$$f(x) = \begin{cases} e^{-1/x^2} & , x \neq 0 \\ x & , x = 0 \end{cases} \qquad \text{whose graph is}$$

Note $f(0) = 0$. With considerable more work we can show that all of f's derivatives at 0 are 0, i.e. $f^{(n)}(0) = 0$. Thus the Taylor series P_∞ about the center $x_0 = 0$ of f is the constant function $P_\infty(x) = 0$ for each $x \in \mathbb{R}$. So $f(x) = P_\infty(x)$ only when $x = x_0$.

Why Theorem 1 is true. Suppose f has a power series representation (expansion) about x_0, i.e, for some $R > 0$

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad \text{valid when} \quad |x - x_0| < R.$$

Since we can differentiate a power series term-by-term within it's interval of convergence, when $|x - x_0| < R$

$$\begin{array}{lllllll}
f(x) & = c_0 + c_1(x-x_0) & + c_2(x-x_0)^2 & + c_3(x-x_0)^3 & + c_4(x-x_0)^4 & + c_5(x-x_0)^5 & + \ldots \\
f^{(1)}(x) & = \quad c_1 & + 2c_2(x-x_0)^1 & + 3c_3(x-x_0)^2 & + 4c_4(x-x_0)^3 & + 5c_5(x-x_0)^4 & + \ldots \\
f^{(2)}(x) & = & 2c_2 & + 2\cdot 3c_3(x-x_0)^1 & + 3\cdot 4c_4(x-x_0)^2 & + 4\cdot 5c_5(x-x_0)^3 & + \ldots \\
f^{(3)}(x) & = & & 2\cdot 3c_3 & + 2\cdot 3\cdot 4c_4(x-x_0)^1 & + 3\cdot 4\cdot 5c_5(x-x_0)^2 & + \ldots \\
f^{(4)}(x) & = & & & 2\cdot 3\cdot 4c_4 & + 2\cdot 3\cdot 4\cdot 5c_5(x-x_0)^1 & + \ldots \\
\vdots & .
\end{array}$$

Thus

$$\begin{array}{llllllll}
f^{(0)}(x_0) & = c_0 + 0 & + 0 & + 0 & + 0 & + 0 & + \ldots & = (0!)\, c_0 \\
f^{(1)}(x_0) & = \quad c_1 & + 0 & + 0 & + 0 & + 0 & + \ldots & = (1!)\, c_1 \\
f^{(2)}(x_0) & = & 2c_2 & + 0 & + 0 & + 0 & + \ldots & = (2!)\, c_2 \\
f^{(3)}(x_0) & = & & 2\cdot 3c_3 & + 0 & + 0 & + \ldots & = (3!)\, c_3 \\
f^{(4)}(x_0) & = & & & 2\cdot 3\cdot 4c_4 & + 0 & + \ldots & = (4!)\, c_4 \\
\vdots & .
\end{array}$$

Notice we get that $f^{(n)}(x_0) = (n!)\, c_n$ and so $c_n = \frac{f^{(n)}(x_0)}{n!}$.

Examples 1-3 from Class Lecture

Example 1. We were given the center $x_0 = 0$ and the function

$$f(x) = \frac{1}{1-x} .$$

We have computed the Taylor series P_∞ as

$$P_\infty(x) = \sum_{n=0}^{\infty} x^n . \tag{1}$$

The power series series in (1) converges when $|x| < 1$, which we can show using the ratio/root test.

Recall that for the Geometric Series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{valid when} \quad |r| < 1,$$

which we showed some time ago by considering the $s_n - rs_n$. Replacing r by x we get

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{valid when} \quad |x| < 1,$$

which gives that the function f has a power series representation.

So by Theorem 1, f's power series representation must be f's Taylor series! So the function $f(x) = \frac{1}{1-x}$ is equal to it's Taylor series $y = P_\infty(x)$ when $|x| < 1$.

Example 2. We were given the center $x_0 = \pi$ and the function

$$f(x) = \sin x .$$

We have computed the Taylor series P_∞ as

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{(-1)^{(n+1)}}{(2n+1)!} (x-\pi)^{2n+1} . \tag{2}$$

The power series in (2) converges for each $x \in \mathbb{R}$, which we can show using the ratio test.

Example 3. We were given the center $x_0 = 0$ and the function

$$f(x) = \ln(1+x) .$$

We have computed the Taylor series P_∞ as

$$P_\infty(x) = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} x^n \tag{3}$$

The power series in (3) converges when $x \in (-1, +1]$. which we show using: ratio test, AST, and p-series.