A main tool used in deciding if a positive-termed series converges is the monotonic sequence theorem. Recall: a sequence \( \{x_n\}_{n \in \mathbb{N}} \) is bounded above means that there is a \( B \in \mathbb{R} \) such that for each \( n \in \mathbb{N} \) \( x_n \leq B \).

**MONOTONIC SEQUENCE THEOREM**

If \( \{x_n\}_{n \in \mathbb{N}} \) is an increasing sequence (i.e., \( x_n \leq x_{n+1} \) for each \( n \in \mathbb{N} \)), then either

1. \( \{x_n\}_{n \in \mathbb{N}} \) is bounded above, in which case \( \lim_{n \to \infty} x_n \) exists (as a finite real number) or
2. \( \{x_n\}_{n \in \mathbb{N}} \) is not bounded above, in which case \( \lim_{n \to \infty} x_n = \infty \).

**KEY IDEA BEHIND POSITIVE-TERMED SERIES**

**Definition:** \( \sum a_n \) is a positive-term series if \( a_n \geq 0 \) for each \( n \).

**Explore:** Let \( \sum a_n \) be a positive-term series.

1. Consider its sequence of partial sums \( \{S_N\}_{N \in \mathbb{N}} \) where \( S_N = a_1 + a_2 + \ldots + a_N \).
2. Recall that \( \sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N \).
3. \( [a_n \geq 0 \text{ for each } n \in \mathbb{N}] \implies [S_N \leq S_{N+1} \text{ for each } N \in \mathbb{N}] \).
4. So \( \{S_N\}_{N \in \mathbb{N}} \) is an increasing sequence.

So either:

1. \( \{S_N\}_{N \in \mathbb{N}} \) is bounded above, in which case, by the monotonic sequence theorem, \( \lim_{N \to \infty} S_N \) exists (as a finite real number) and thus by (2) above, \( \sum a_n \) converges (to a finite real number)
2. \( \{S_N\}_{N \in \mathbb{N}} \) is not bounded above, in which case, by the monotonic sequence theorem, \( \lim_{N \to \infty} S_N = \infty \) and thus by (2) above, \( \sum a_n \) diverges (to \( \infty \)).

**POSITIVE-TERMED SERIES CRITERIA**

Let \( \sum a_n \) be a positive-term series. Set

\[
S_N := \sum_{n=1}^{N} a_n ,
\]

Then either

1. there exists a bound \( B \) so that for each \( N \) we have \( s_N \leq B \), in which case, the series \( \sum a_n \) converges (to a finite real number)
2. \( \lim_{N \to \infty} s_N = \infty \), in which case, the series \( \sum a_n \) diverges (to \( \infty \)).