0. Fill-in-the boxes. All series \sum and \sum_n are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

Positive-Termed Series Criteria (so for $\sum a_n$ where $a_n \ge 0$)

Let $\sum a_n$ be a positive-termed series. We consider its sequence of partial sums $\{s_n\}_n$ where

$$s_n \stackrel{\text{def}}{:=} \sum_{k=1}^n a_k \; .$$

The behavior of series $\sum_{n} a_n$ is, by definition, the same as the behavoir of sequence $\{s_n\}_n$. The key observation is that, because $a_n \ge 0$, the sequence $\{s_n\}_n$ is increasing (i.e., $s_n \le s_{n+1}$). So: either

• the sequence of partial sums $\{s_n\}_n$ is bounded above, i.e., there is some big number B so that for each n we have that $s_n \leq B$, in which case, the series $\sum a_n$ converges (to a finite real number)

or

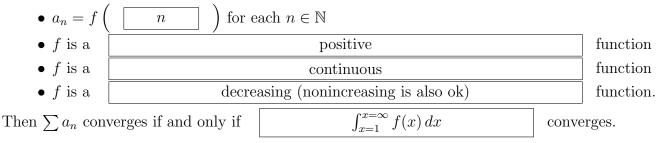
• $\lim_{n \to \infty} s_n = \infty$,

the series $\sum a_n$ diverges (to ∞).



0a. State the **Integral Test** for a positive-termed series $\sum a_n$.

Let $f: [1, \infty) \to \mathbb{R}$ be so that



0b. State the **Direct Comparison Test (DCT)** for a positive-termed series $\sum a_n$. Let $N_0 \in \mathbb{N}$.

• If $0 \le a_n \le c_n$ when $n \ge N_0$ and $\sum c_n$ converges , then $\sum a_n$ converges. • If $0 \le d_n \le a_n$ when $n \ge N_0$ and $\sum d_n$ diverges , then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0c. State the Limit Comparison Test (LCT) for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n \to \infty} \boxed{\frac{a_n}{b_n}}$. • If $0 < L < \infty$, then $[\sum b_n \text{ converges } \iff \sum a_n \text{ converges }]$. • If L = 0, then $[\sum b_n \text{ converges } \implies \sum a_n \text{ converges }]$. • If $L = \infty$, then $[\sum b_n \text{ diverges } \implies \sum a_n \text{ diverges }]$.

Goal: cleverly pick positive b_n 's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

Helpful Intuition

<u>Claim 1</u>: If x > 0, then

 $\ln x \leq x^1 \leq e^x .$ To see this, consider the function $g(x) = e^x - x$. Then g(0) = 1 and $g'(x) = e^x > 0$ for x > 0. So for all x > 0, we have g(x) > 0, i.e., $e^x - x > 0$. So $e^x \geq x^1$. Recall that the graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ over the line y = x.

<u>Claim 2</u>: Consider a positive power q > 0. There is (some big number) $N_q > 0$ so that if $x \ge N_q$ then

$$\boxed{\ln x \le x^q \le e^x} \quad .$$

To see Claim 2, use L'Hôpital's rule to show that

$$\lim_{x \to \infty} \frac{\log_e x}{x^q} = 0 \qquad \text{and} \qquad \lim_{x \to \infty} \frac{x^q}{e^x} = 0. \qquad (*)$$

<u>Claim 3</u>: Consider a positive power q > 0 along with a base b > 1. There is (some big #) $N_{q,b} > 0$ so that if $x \ge N_{q,b}$ then

$$\log_b x \le x^q \le b^x$$

To see Claim 3, recall that $\log_e x = \ln x$. Recall that for any base b > 0 with $b \neq 1$

$$\log_b x = \frac{\log_e x}{\log_e b}$$
 and $D_x \log_b x = \frac{1}{x \ln b}$ and $D_x b^x = b^x \ln b$

and $\lim_{x\to\infty} b^x = \infty$ if and only if b > 1. And so (*) holds if one replaces e with any base b > 1. <u>Moral</u>: To figure out what is happening to a series involving $\log_b n$ or b^n , keep in mind that as $n \to \infty$

- $\log_b n$ grows super slow compared to n^q
- b^n grows super fast compared to n^q

for any positive power q > 0 and base b > 1.