0. Fill-in-the boxes. All series $\sum$ and $\sum_n$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

Positive-Termed Series Criteria
(so for $\sum a_n$ where $a_n \geq 0$)

Let $\sum a_n$ be a positive-termed series. We consider its sequence of partial sums $\{s_n\}_n$ where

$$s_n \overset{\text{def}}{:=} \sum_{k=1}^{n} a_k \ .$$

The behavior of series $\sum_n a_n$ is, by definition, the same as the behavoir of sequence $\{s_n\}_n$.
The key observation is that, because $a_n \geq 0$, the sequence $\{s_n\}_n$ is increasing (i.e., $s_n \leq s_{n+1}$). So: either

- the sequence of partial sums $\{s_n\}_n$ is bounded above, i.e., there is some big number B so that for each n we have that $s_n \leq B$, in which case, the series $\sum a_n$ converges (to a finite real number)

or

- $\lim_{n\to\infty} s_n = \infty$, in which case, the series $\sum a_n$ diverges (to ∞).

Tests for Positive-Termed Series
(so for $\sum a_n$ where $a_n \geq 0$)

0a. State the **Integral Test** for a positive-termed series $\sum a_n$.
Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

- $a_n = f\left(\boxed{n} \right)$ for each $n \in \mathbb{N}$
- f is a $\boxed{\text{positive}}$ function
- f is a $\boxed{\text{continuous}}$ function
- f is a $\boxed{\text{decreasing (nonincreasing is also ok)}}$ function.

Then $\sum a_n$ converges if and only if $\boxed{\int_{x=1}^{x=\infty} f(x)\,dx}$ converges.

0b. State the **Direct Comparison Test (DCT)** for a positive-termed series $\sum a_n$.
Let $N_0 \in \mathbb{N}$.

- If $\boxed{0 \leq a_n \leq c_n}$ when $n \geq N_0$ and $\boxed{\sum c_n \text{ converges}}$, then $\sum a_n$ converges.
- If $\boxed{0 \leq d_n \leq a_n}$ when $n \geq N_0$ and $\boxed{\sum d_n \text{ diverges}}$, then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0c. State the **Limit Comparison Test (LCT)** for a positive-termed series $\sum a_n$.
Let $b_n > 0$ and $L = \lim_{n\to\infty} \boxed{\frac{a_n}{b_n}}$.

- If $\boxed{0 < L < \infty}$, then $\boxed{[\sum b_n \text{ converges} \iff \sum a_n \text{ converges}]}$.
- If $\boxed{L = 0}$, then $\boxed{[\sum b_n \text{ converges} \implies \sum a_n \text{ converges}]}$.
- If $\boxed{L = \infty}$, then $\boxed{[\sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}]}$.

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

Helpful Intuition

Claim 1: If $x > 0$, then

$$\ln x \le x^1 \le e^x \ .$$

To see this, consider the function $g(x) = e^x - x$. Then $g(0) = 1$ and $g'(x) = e^x > 0$ for $x > 0$. So for all $x > 0$, we have $g(x) > 0$, i.e., $e^x - x > 0$. So $e^x \ge x^1$.
Recall that the graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ over the line $y = x$.

Claim 2: Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \ge N_q$ then

$$\boxed{\boxed{\ln x \le x^q \le e^x}} \ .$$

To see Claim 2, use L'Hôpital's rule to show that

$$\lim_{x\to\infty} \frac{\log_e x}{x^q} \ = \ 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{x^q}{e^x} \ = \ 0 \ . \qquad (*)$$

Claim 3: Consider a positive power $q > 0$ along with a base $b > 1$.
There is (some big #) $N_{q,b} > 0$ so that if $x \ge N_{q,b}$ then

$$\log_b x \le x^q \le b^x$$

To see Claim 3, recall that $\log_e x = \ln x$. Recall that for any base $b > 0$ with $b \ne 1$

$$\log_b x = \frac{\log_e x}{\log_e b} \qquad \text{and} \qquad D_x \log_b x = \frac{1}{x \ln b} \qquad \text{and} \qquad D_x b^x = b^x \ln b$$

and $\lim_{x\to\infty} b^x = \infty$ if and only if $b > 1$. And so $(*)$ holds if one replaces e with any base $b > 1$.

Moral: To figure out what is happening to a series involving $\log_b n$ or b^n, keep in mind that as $n \to \infty$

- $\log_b n$ grows *super slow* compared to n^q
- b^n grows *super fast* compared to n^q

for any positive power $q > 0$ and base $b > 1$.