**0.** Fill-in-the boxes. All series  $\sum \text{ and } \sum_n$  are understood to be  $\sum_{n=1}^{\infty}$ , unless otherwise indicated.

**Positive-Termed Series Criteria**  
(so for 
$$\sum a_n$$
 where  $a_n \ge 0$ )

Let  $\sum a_n$  be a positive-termed series. We consider its sequence of partial sums  $\{s_n\}_n$  where

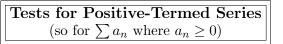
$$s_n \stackrel{\text{def}}{:=} \sum_{k=1}^n a_k \; .$$

The behavior of series  $\sum_{n} a_n$  is, by definition, the same as the behavior of sequence  $\{s_n\}_n$ . The key observation is that, because  $a_n \ge 0$ , the sequence  $\{s_n\}_n$  is increasing (i.e.,  $s_n \le s_{n+1}$ ). So: either

• the sequence of partial sums  $\{s_n\}_n$  is bounded above, i.e., there is some big number B so that for each n we have that  $s_n \leq B$ , in which case, the series  $\sum a_n$  converges (to a <u>finite</u> real number)

or

•  $\lim_{n \to \infty} s_n = \infty$ , the series  $\sum a_n$  diverges (to  $\infty$ ).



## **0a.** State the **Integral Test** for a positive-termed series $\sum a_n$ .

Let  $f: [1,\infty) \to \mathbb{R}$  be so that

• $a_n = f\left( \right)$		$\left[ \right]$ for each $\left[ \right]$	$ach \ n \in \mathbb{N}$	4			
• $f$ is a		,					function
• $f$ is a							function
• $f$ is a							function.
Then $\sum a_n$ converges if and only if $\square$							erges.
<b>0b.</b> State the <b>Direct Comparison Test (DCT)</b> for a positive-termed series $\sum a_n$ . Let $N_0 \in \mathbb{N}$ .							
• If		when $n \ge$	$\geq N_0$ and		, the	en $\sum a$	$a_n$ converges.
• If		when $n \geq$	$\geq N_0$ and		, the	en $\sum a$	$u_n$ diverges.
Hint: sing the song to yourself.							
<b>0c.</b> State the Limit Comparison Test (LCT) for a positive-termed series $\sum a_n$ .							
Let $b_n > 0$ and	$L = \lim_{n \to \infty} $	)					
• If		,	then				
• If		,	then				
• If		,	then				
Goal: cleverly pick positive $b_n$ 's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_{n}$ converges.							

in which case,

## Helpful Intuition

<u>Claim 1</u>: If x > 0, then

 $\ln x \leq x^1 \leq e^x .$ To see this, consider the function  $g(x) = e^x - x$ . Then g(0) = 1 and  $g'(x) = e^x > 0$  for x > 0. So for all x > 0, we have g(x) > 0, i.e.,  $e^x - x > 0$ . So  $e^x \geq x^1$ . Recall that the graph of  $y = \ln x$  is the reflection of the graph of  $y = e^x$  over the line y = x.

<u>Claim 2</u>: Consider a positive power q > 0. There is (some big number)  $N_q > 0$  so that if  $x \ge N_q$  then

$$\boxed{\ln x \le x^q \le e^x} \quad .$$

To see Claim 2, use L'Hôpital's rule to show that

$$\lim_{x \to \infty} \frac{\log_e x}{x^q} = 0 \qquad \text{and} \qquad \lim_{x \to \infty} \frac{x^q}{e^x} = 0. \qquad (*)$$

<u>Claim 3</u>: Consider a positive power q > 0 along with a base b > 1. There is (some big #)  $N_{q,b} > 0$  so that if  $x \ge N_{q,b}$  then

$$\log_b x \le x^q \le b^x$$

To see Claim 3, recall that  $\log_e x = \ln x$ . Recall that for any base b > 0 with  $b \neq 1$ 

$$\log_b x = \frac{\log_e x}{\log_e b}$$
 and  $D_x \log_b x = \frac{1}{x \ln b}$  and  $D_x b^x = b^x \ln b$ 

and  $\lim_{x\to\infty} b^x = \infty$  if and only if b > 1. And so (\*) holds if one replaces e with any base b > 1. <u>Moral</u>: To figure out what is happening to a series involving  $\log_b n$  or  $b^n$ , keep in mind that as  $n \to \infty$ 

- $\log_b n$  grows super slow compared to  $n^q$
- $b^n$  grows super fast compared to  $n^q$

for any positive power q > 0 and base b > 1.