Polar Coordinates

Our old trustly friend, Cartesian coordinates, are handy when dealing with *boxy* objects.
Our new friend, polar coordinates, are handy when dealing with *windy/circular* objects.
In this handout, let's abbreviate:

Cartesian coordinates by CC and *polar coordinates* by PC .

Basics

Let's start with a point $P \in \mathbb{R}^2$. Then P has a unique CC representation (x, y).
DEFINITION A representation of this point P in polar coordinates is any (r, θ) where

$$x = r \cos\theta \qquad \text{and} \qquad y = r \sin\theta .$$

Given an (x, y), how are you going to find such an (r, θ)? Let's start by asking Mr. Happy Unit Circle.
Next, some useful observations.

- When working in CC, $[\,(x, y) = (\tilde{x}, \tilde{y})\,]$ if and only if $[\,x = \tilde{x}$ and $y = \tilde{y}\,]$.
- If the point P has PC (r, θ), then P also has PC $(r, \theta + 2\pi)$. In other word, in PC,

  (r, θ) represents the same point as $(r, \theta + 2\pi)$.

  This is because the point P has the unique CC (x, y) where

$$x = r\cos\theta \overset{\text{note}}{=} r\cos(\theta + 2\pi)$$
$$y = r\sin\theta \overset{\text{note}}{=} r\sin(\theta + 2\pi) .$$

- If the point P has PC $(-r, \theta)$, then P also has PC $(r, \theta + \pi)$. In other word, in PC,

  $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

  This is because the point P has the unique CC (x, y) where[1]

$$x = -r\cos\theta \overset{\text{note}}{=} +r\cos(\theta + \pi)$$
$$y = -r\sin\theta \overset{\text{note}}{=} +r\sin(\theta + \pi) .$$

Conversion

A point $P \in \mathbb{R}^2$ with CC (x, y) and PC (r, θ) satisfies the following.
By definition of polar coordinates:

$$x = r \cos\theta \qquad \text{and} \qquad y = r \sin\theta . \tag{1}$$

And so by basic trigonometry:

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 . \end{cases}$$

So is given a point P in PC (r, θ), we can find it's (unique) CC (x, y) by using the equation (1).
While if given a point P in CC (x, y), how to find a PC (r, θ)? ...
There are so many choices. Well, e.g.: we can use:[2]

$$r = \sqrt[+]{x^2 + y^2} \qquad \text{and} \qquad \theta = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ \frac{-\pi}{2} & \text{if } x = 0 \text{ and } y < 0 , \end{cases}$$

which gives $r \geq 0$ and $\frac{-\pi}{2} \leq \theta < \frac{3\pi}{2}$. Can you think of other choices?

[1] Recall $\cos(\theta + \pi) = -\cos\theta$ and $\sin(\theta + \pi) = -\sin\theta$.
[2] Recall, $\frac{-\pi}{2} < \arctan\theta < \frac{\pi}{2}$.

Polar Equations

Consider a polar equation $r = f(\theta)$. You can think of such a polar equation as a describing a *parametric curve* given in CC by (use equations in (1)),

$$\begin{aligned} x(\theta) &= f(\theta)\cos\theta \\ y(\theta) &= f(\theta)\sin\theta\ . \end{aligned} \tag{2}$$

Graphing Polar equation $r = f(\theta)$

The period of $f(\theta) = \cos(k\theta)$ and of $f(\theta) = \sin(k\theta)$ is $\frac{2\pi}{k}$.
To sketch these graphs, divide the period by 4 and make *the chart*.
We divide the period by 4 when making *the chart* in order to detect the max/min/zero's of the function $r = f(\theta)$.

Area

Let $A(r,\theta)$ be the area of a sector of a circle with radius r and cental angle θ radians.
Comparing $A(r,\theta)$ to the area of the whole circle lead us to a proportion, which we can solve for $A(r,\theta)$:

$$\frac{A(r,\theta)}{A(r,2\pi)} = \frac{\theta}{2\pi} \implies \frac{A(r,\theta)}{\pi r^2} = \frac{\theta}{2\pi} \implies A(r,\theta) = \frac{\theta}{2\pi}\,\frac{\pi r^2}{1} \implies A(r,\theta) = \frac{\theta r^2}{2}\ .$$

So, the area of a sector of a circle with radius r and central angle $\Delta\theta$ is

$$A(r,\Delta\theta) = \frac{1}{2} r^2\,(\Delta\theta)\ .$$

Now consider a function $r = f(\theta)$ which determines a curve in the plane where

(1) $f\colon [\alpha,\beta] \to [0,\infty]$
(2) f is continuous on $[\alpha,\beta]$
(3) $\beta - \alpha \le 2\pi$.

Then the area bounded by polar curves $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} [f(\theta)]^2\, d\theta\ .$$

Arc Length

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r,\theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the arc) length of the curve is

$$\text{AL} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta\ .$$

Why is the so? Well, veiwing the curve that $r = f(\theta)$ traces out as a *parametric curve* as given in (2), we already know that

$$\text{AL} = \int_\alpha^\beta \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta\ .$$

And

$$\begin{aligned} \left[x'(\theta)\right]^2 + \left[y'(\theta)\right]^2 &= [D_\theta(f(\theta)\cos\theta)]^2 + [D_\theta(f(\theta)\sin\theta)]^2 \\ &= \left[{}^-f(\theta)\sin\theta + f'(\theta)\cos\theta\right]^2 + \left[{}^+f(\theta)\cos\theta + f'(\theta)\cos\theta\right]^2 \\ &= \quad [f(\theta)]^2\sin^2\theta - 2f(\theta)f'(\theta)\cos\theta\sin\theta + \left[f'(\theta)\right]^2\cos^2\theta \\ &\quad + [f(\theta)]^2\cos^2\theta + 2f(\theta)f'(\theta)\cos\theta\sin\theta + \left[f'(\theta)\right]^2\sin^2\theta \\ &= [f(\theta)]^2\left(\sin^2\theta + \cos^2\theta\right) + \left[f'(\theta)\right]^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= [f(\theta)]^2 + \left[f'(\theta)\right]^2 \\ &= [r]^2 + \left[\frac{dr}{d\theta}\right]^2\ . \end{aligned}$$