If $y = f(x)$ and $y = g(x)$ are polynomials, then it follows from a theorem in algebra that

$$\underbrace{\frac{f(x)}{g(x)}}_{\textbf{rational function}} = \underbrace{P(x)}_{\text{a polynomial}} + \underbrace{F_1(x) + F_2(x) + \ldots + F_k(x)}_{\textbf{partial fraction decomposition}}, \tag{1}$$

where each **partial fraction** F_i has one of the forms

$$\frac{A}{(px+q)^m} \qquad \text{or} \qquad \frac{Cx+D}{(ax^2+bx+c)^n}$$

where

- m and n are nonnegative integers, i.e., $n, m \in \{0, 1, 2, 3, 4, 5, \ldots\}$
- $ax^2 + bx + c$ is irreducible, i.e., it cannot be factored over $\mathbb{R}$, i.e. $b^2 - 4ac < 0$.

Why do we care? Well, if (1) holds then

$$\underbrace{\int \frac{f(x)}{g(x)}\,dx}_{\textbf{we want to find this}} = \underbrace{\int P(x)\,dx}_{\text{easy to find}} + \underbrace{\int [F_1(x) + F_2(x) + \ldots + F_k(x)]\,dx}_{\textbf{do-able}} .$$

So how to find this decomposition ....

First Case: [degree of $y = f(x)$] $<$ [degree of $y = g(x)$]

In this case, $P(x) = 0$ in (1). Express $y = g(x)$ as a product of

- linear factors $px + q$
- *irreducible* quadratic factors $ax^2 + bx + c$ (irreducible means that $b^2 - 4ac < 0$).

Collect up the repeated factors so that $g(x)$ is a product of *different* factors of the form $(px+q)^m$ and $(ax^2+bx+c)^n$. Then apply the following rules.

Linear Rule: For each factor of the form $(px+q)^m$ where $m \geq 1$, the decomposition (1) contains a sum of m partial factions of the form

$$\frac{A_1}{(px+q)^1} + \frac{A_2}{(px+q)^2} + \ldots + + \frac{A_m}{(px+q)^m}$$

where each A_i is a real number.

IQ[1] **Rule**: For each factor of the form $(ax^2+bx+c)^n$ where $n \geq 1$ and $b^2 - 4ac < 0$, the decomposition (1) contains a sum of n partial factions of the form

$$\frac{A_1x+B_1}{(ax^2+bx+c)^1} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where the A_i's and B_i's are real number.

[1]**IQ** stands for *irreducible quadratic*.

Second Case: [degree of $y = f(x)$] $\geq$ [degree of $y = g(x)$]

First do long division to express $\frac{f(x)}{g(x)}$ as

$$\frac{f(x)}{g(x)} = \underbrace{P(x)}_{\text{a polynomial}} + \underbrace{\frac{R(x)}{g(x)}}_{[\text{degree of } y=R(x)] \; < \; [\text{degree of } y=g(x)]} ,$$

How to do this? Well we surely see that

$$\frac{5}{3} = 1\,\frac{2}{3} = 1 + \frac{2}{3} \,;$$

we get this by long division

$$\begin{array}{r} 1 \\ 3\overline{)5} \\ \\ 3 \\ \hline \\ 2\,. \end{array}$$

Similarly,

$$\frac{f(x)}{g(x)} = P(x) + \frac{R(x)}{g(x)} ,$$

where

$$\begin{array}{r} \mathrm{P(x)} \\ g(x)\overline{)f(x)} \\ \\ \vdots \\ \hline \\ R(x)\,. \end{array}$$

Now we can apply the **First Case** to $\frac{R(x)}{g(x)}$ since [degree of $y = R(x)$] $<$ [degree of $y = g(x)$].

A common mistake when have x^2 in the denominator. Note that

$$x^2 = (x-0)^2 = 1x^2 + 0x + 0$$

and so $b^2 - 4ac = 0 \not< 0$. So we follow the **Linear Rule** to see that the partial fraction decomposition of $\frac{1}{x^2}$ is of the form

$$\frac{1}{x^2} = \frac{A}{x^1} + \frac{B}{x^2} .$$

Note that $A = 0$ and $B = 1$. A common mistake is to try to use **IQ Rule**, which would give

$$\frac{1}{x^2} \overset{\text{wrong}}{=} \frac{Ex+F}{x^1} + \frac{Gx+H}{x^2} .$$

This would still lead to the correct answer ($E = F = G = 0$ and $H = 1$) but you have to do LOTS of work to get to it.