A rational function $y = \frac{f(x)}{g(x)}$ (recall rational means that f and g are polyominals)
has a **Partial Faction Decomposition (PFD)**

$$\underbrace{\frac{f(x)}{g(x)}}_{\textbf{rational function}} = \underbrace{P(x)}_{\text{a polynomial}} + \underbrace{F_1(x) + F_2(x) + \ldots + F_k(x)}_{\textbf{partial fractions}}, \tag{PFD}$$

where each **partial fraction** F_i has one of the forms

$$\frac{A}{(px+q)^m} \qquad \text{or} \qquad \frac{Cx+D}{(ax^2+bx+c)^n}$$

where

- $p \neq 0$ and $a \neq 0$
- m and n are integers, i.e., $n, m \in \mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$
- ax^2+bx+c is irreducible (i.e. cannot be factored) over $\mathbb{R}$, (now think quad. formula) i.e., $b^2-4ac < 0$.

Why do we care? Well, if we can find the (PFD), then

$$\underbrace{\int \frac{f(x)}{g(x)}\,dx}_{\textbf{we want to find this}} = \underbrace{\int P(x)\,dx}_{\text{easy to find}} + \underbrace{\int F_1(x)\,dx + \int F_2(x)\,dx + \ldots + \int F_k(x)\,dx}_{\textbf{each of these integral is do-able by previously learned methods}} .$$

So how to find this PFD ....

$$\boxed{\text{First Case:} \qquad [\text{degree of } y = f(x)] \; < \; [\text{degree of } y = g(x)]}$$

In this case, $P(x) = 0$ in (PFD). Begin by expressing the denominator $y = g(x)$ as a product of:

- linear factors $px + q$
- *irreducible* quadratic factors $ax^2 + bx + c$ (irreducible means that $b^2 - 4ac < 0$).

Collect up the repeated factors so that g is a product of *different* factors of the form $(px + q)^m$ and $(ax^2 + bx + c)^n$. Then apply the following rules.

Linear Rule: For each linear factor of the form $(px + q)^m$,
the (PFD) contains a sum of m partial fractions of the form

$$\frac{A_1}{(px+q)^1} + \frac{A_2}{(px+q)^2} + \ldots + + \frac{A_m}{(px+q)^m}$$

where each A_i is a real number.

IQ[1] **Rule**: For each IQ (so $b^2 - 4ac < 0$) factor of the form $(ax^2 + bx + c)^n$,
the (PFD) contains a sum of n partial factions of the form

$$\frac{A_1x+B_1}{(ax^2+bx+c)^1} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where the A_i's and B_i's are real number.

[1] **IQ** stands for *irreducible quadratic.*

$$\boxed{\text{Second Case:} \qquad [\text{degree of } y = f(x)] \;\geq\; [\text{degree of } y = g(x)]}$$

First do long division to express $\frac{f(x)}{g(x)}$ as

$$\frac{f(x)}{g(x)} = \underbrace{P(x)}_{\text{a polynomial}} + \underbrace{\frac{R(x)}{g(x)}}_{[\text{degree of } y=R(x)] \,<\, [\text{degree of } y=g(x)]} ,$$

How to do this? Well we surely see that

$$\frac{5}{3} = 1\,\frac{2}{3} = 1 + \frac{2}{3} \,;$$

we get this by long division

$$\begin{array}{r} 1 \\ 3\overline{)5} \\ 3 \\ \hline 2\,. \end{array}$$

Similarly,

$$\frac{f(x)}{g(x)} = P(x) + \frac{R(x)}{g(x)} \,,$$

where

$$\begin{array}{r} \mathrm{P(x)} \\ g(x)\overline{)f(x)} \\ \vdots \\ \hline R(x)\,. \end{array}$$

Now we can apply the **First Case** to $\frac{R(x)}{g(x)}$ since $[\text{degree of } y = R(x)] \;<\; [\text{degree of } y = g(x)]$.

<u>A common mistake when have x^2 in the denominator.</u> Note that

$$x^2 = (x-0)^2 = 1x^2 + 0x + 0$$

and so $b^2 - 4ac = 0 \not< 0$. So we follow the **Linear Rule** to see that the partial fraction decomposition of $\frac{1}{x^2}$ is of the form

$$\frac{1}{x^2} = \frac{A}{x^1} + \frac{B}{x^2} \,.$$

Note that $A = 0$ and $B = 1$. A common mistake is to try to use **IQ Rule**, which would give

$$\frac{1}{x^2} \overset{\text{wrong}}{=} \frac{Ex+F}{x^1} + \frac{Gx+H}{x^2} \,.$$

This would still lead to the correct answer ($E = F = G = 0$ and $H = 1$) but you have to do LOTS of work to get to it.

PFD

Ex $\int \frac{x^3-4x-1}{x(x-1)^3}\,dx$

- strictly bigger bottom? Yes/No $[\text{deg dem.}] = 4 \overset{?}{>} [\text{deg. num.}] = 3$
- $x = (x-0)^1 = (\text{linear term})^1$ ← contribute 1 factor
- $(x-1)^3 = (\text{linear term})^3$ ← contribute 3 factors

$$\frac{x^3-4x-1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \underset{\text{LCD}}{\overset{\text{add}}{=}} \frac{A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx}{x(x-1)^3}$$

in PFD, should always be the same

plug in the zeros of the linear terms

equate numerators

$$x^3-4x-1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

$x=0 \Rightarrow -1 = -A \Rightarrow \boxed{A=1}$

$x=1 \Rightarrow \boxed{-4=D}$

Long Way → $x^3-4x-1 = A(x^3-3x^2+3x-1) + B(x^3-2x^2+x) + C(x^2-x) + Dx$

Short Way

Equate coefficients – long way (multiply out & collect like terms) or short way.

$$x^3-4x-1 = Ax^3 - 3Ax^2 + 3Ax - A + Bx^3 - 2Bx^2 + Bx + Cx^2 - Cx + Dx$$

$$1x^3 - 0x^2 - 4x - 1x^0 = (A+B)x^3 + (-3A-2B+C)x^2 + (3A+B-C+D)x - A$$

$\Rightarrow$ x^3 : $1 = A+B \longrightarrow 1 = 1+B \Rightarrow \boxed{B=0}$

x^2 : $0 = -3A-2B+C \longrightarrow 0 = -3(1)-2(0)+C \Rightarrow \boxed{C=3}$

x^1 : $-4 = 3A+B-C+D$

constant : $-1 = -A$

solve for constants

So

$$\int \frac{x^3-4x-1}{x(x-1)^3}\,dx = \int \left[\frac{1}{x} + \frac{0}{x-1} + \frac{3}{(x-1)^2} + \frac{-4}{(x-1)^3}\right] dx$$

$$= \int \frac{dx}{x} + 3\int (x-1)^{-2}\,dx - 4\int (x-1)^{-3}\,dx$$

$$= \ln|x| + \frac{3(x-1)^{-1}}{-1} - 4\,\frac{(x-1)^{-2}}{-2} + K$$

we already used C

$$= \boxed{\ln|x| - \frac{3}{x-1} + \frac{2}{(x-1)^2} + K}$$

PFD

Ex. $\displaystyle\int \frac{(4x+1)}{4x^2-12x+34}\,dx$

PFD

- strictly bigger bottom? (deg dem.) $= 2 \; > \;$ (deg num) $= 1$ Yes
- $b^2-4ac = (-12)^2 - 4(4)(34) = -400 < 0 \Rightarrow 4x^2-12x+34 = (\text{irred. quad})^1$
- $\dfrac{4x+1}{4x^2-12x+34} \underset{\text{looks like}}{\overset{\text{PFD}}{=}} \dfrac{Ax+B}{4x^2-12x+34}$ Easy to find A & B: $A=4$ & $B=1$
- So $\dfrac{4x+1}{4x^2-12x+34}$ is already in it's PFD. So to $\int$, use previous method
- Note $4x^2-12x+34 \underset{\text{square}}{\overset{\text{complete}}{=}} (2x-3)^2+25$
- Let $\boxed{t = 4x^2-12x+34}$ and $\boxed{u = 2x-3}$
- Goal $\displaystyle\int \frac{4x+1}{4x^2-12x+34}\,dx = (\text{a constant})\int \frac{dt}{t} + (\text{a constant})\int \frac{du}{u^2 \pm a^2}$

$$\int \frac{4x+1}{4x^2-12x+34}\,dx = \int \frac{\boxed{?}\,\boxed{(8x-12)\,dx}_{\leftarrow dt}}{4x^2-12x+34} + \int \frac{\boxed{?}\,dx}{4x^2-12x+34}$$

$$= \int \frac{\overbrace{\left(\tfrac{1}{2}\right)\boxed{(8x-12)\,dx}}^{4x-6}}{4x^2-12x+34} + \int \frac{7\,dx}{4x^2-12x+34}$$

$$= \frac{1}{2}\int \frac{\boxed{(8x-12)\,dx}_{\leftarrow dt}}{4x^2-12x+34} + 7\left(\tfrac{1}{2}\right)\int \frac{\boxed{2\,dx}_{\leftarrow du}}{(2x-3)^2+25}$$

$$= \frac{1}{2}\int \frac{dt}{t} + \frac{7}{2}\int \frac{du}{u^2+5^2}$$

$$= \boxed{\frac{1}{2}\ln\left|4x^2-12x+34\right| + \frac{7}{2}\left(\frac{1}{5}\right)\tan^{-1}\left(\frac{2x-3}{5}\right) + C}$$