The plane (i.e., the 2-dimensional space) $\mathbb{R}^2$ is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

Definition

Given:

(1) an interval $I \subset \mathbb{R}$
(2) a function $f \colon I \to \mathbb{R}$
(3) another function $g \colon I \to \mathbb{R}$.

Then we form

(4) the function $h \colon I \to \mathbb{R}^2$ by letting $h(t) = (f(t), g(t))$ for $t \in I$.

So for a fixed number $t_0 \in I \subset \mathbb{R}$, the point $h(t_0) = (f(t_0), g(t_0)) \in \mathbb{R}^2$. We call

$$\mathcal{C} = \left\{(f(t), g(t)) \in \mathbb{R}^2 \colon t \in I\right\}$$

a <u>paramteric (planar) curve</u>, which is <u>parametrized</u> by the functions f and g. Often we write as

$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned} \qquad , t \in I$$

or write as

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \qquad , t \in I$$

►. We think of t as time and $\mathcal{C}$ describing the motion of a puffo as he moves through the plane $\mathbb{R}^2$.

Calculus with Parametric Curves

Consider the curve $\mathcal{C}$ parameterized by

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

for $a \le t \le b$.

1) Express $\frac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\frac{dy}{dx} = \boxed{\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$

2) The tangent line to $\mathcal{C}$ when $t = t_0$ is $y = mx + b$ where m is $\boxed{\frac{dy}{dx}}$ evaluated at $t = t_0$.

3) Express $\frac{d^2y}{dx^2}$ using derivatives with respect to t. Answer: $\frac{d^2y}{dx^2} = \boxed{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$

4) The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

$$\text{Arc Length} = \boxed{\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt}$$