A. Power Series

Consider the (formal) power series

$$h(x) \; = \; \sum_{n=0}^{\infty} a_n \, (x - x_0)^n \; , \tag{1}$$

with radius of convergence $R \in [0, \infty]$.
(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)
Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

A.1. The choices for nest 4 boxes are: AC, CC, DIVG, anything. Here,

AC stands for: *is always absolutely convergent*
CC stands for: *is always conditionally convergent*
DIVG stands for: *is always divergent*
anything stands for: *can do anything, i.e., there are examples showing that it can be AC, CC, or DIVG.*

(1) At the center $x = x_0$, the power series in (1) $\boxed{\text{AC}}$.
(2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series in (1) $\boxed{\text{AC}}$.
(3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$, the power series in (1) $\boxed{\text{DIVG}}$.
(4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series in (1) $\boxed{\text{anything}}$.

A.2. Now let $R > 0$ and fill-in the 7 boxes.
Consider the function $y = h(x)$ defined by the power series in (1).

(a) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Also, on this interval

$$h'(x) \; = \; \sum_{n \, = \, \boxed{1}}^{\infty} \boxed{n \, a_n \, (x - x_0)^{n-1}} \; . \tag{2}$$

What can you say about the radius of convergence of the power series in (2)?
$\boxed{\text{The power series in (2) has the same raduis of convergence as the power series in (1).}}$.

(b) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x) \, dx \; = \; \sum_{n \, = \, \boxed{0}}^{\infty} \boxed{\frac{a_n}{n+1} \, (x - x_0)^{n+1}} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} \; .$$

B. Taylor/Maclaurin Polynomials and Series

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.
Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

B.1. The formula for c_n is

$$c_n \; = \; \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

B.2. In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N}$$

B.3. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_N(x) = \boxed{\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\ (x-x_0)^n}$$

B.4. In open form (i.e., with ... and without a $\sum$-sign)

$$P_\infty(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots}$$

B.5. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\ (x-x_0)^n}$$

B.6. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}\ (x-x_0)^{(N+1)}}$ for some c between $\boxed{x}$ and $\boxed{x_0}$.

B.7. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.

C. Commonly Used Taylor Series

Here, *expansion* refers to the power series expansion that is the Maclaurin series.

C.1. An expansion for $y = e^x$ is $\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

C.2. An expansion for $y = \cos x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

C.3. An expansion for $y = \sin x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

C.4. An expansion for $y = \frac{1}{1-x}$ is $\boxed{\sum_{n=0}^{\infty} x^n}$, which is valid precisely when $x \in \boxed{(-1, 1)}$.

C.5. An expansion for $y = \ln(1+x)$ is $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$, which is valid precisely when $x \in \boxed{(-1, 1]}$.

C.6. An expansion for $y = \arctan x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$, which is valid precisely when $x \in \boxed{[-1, 1]}$.