0. Fill-in-the boxes.

Power Series Consider the (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n\,(x - x_0)^n \;, \tag{1}$$

with radius of convergence $R \in [0, \infty]$.

(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

0.1. First let $0 < R < \infty$. The largest set of x's for which we know that the power series in (1) is:

(a) absolutely convergent is $(x_0 - R, x_0 + R)$, also ok: $\{x \in \mathbb{R}\colon\; |x - x_0| < R\}$

(b) divergent is $(-\infty, x_0 - R) \cup (x_0 + R, \infty)$, also ok: $\{x \in \mathbb{R}\colon\; |x - x_0| > R\}$.

What can you say about the convergence of the power series in (1) when $x = x_0 + R$ or $x = x_0 - R$?

the series can be doing anything, i.e., there are examples showing that it can be absolutely convergent, conditionally convergent or divergent

0.2. Now let $R = \infty$. The largest set of x's for which we know that the power series in (1) is:

(a) absolutely convergent is $\mathbb{R}$, also ok: $\{x \in \mathbb{R}\colon\; |x - x_0| < R\}$

(b) divergent is $\emptyset$, also ok: the empty set .

0.3. Now let $R = 0$. The largest set of x's for which we know that the power series in (1) is:

(a) absolutely convergent is $\{x_0\}$, also ok: $\{x \in \mathbb{R}\colon x = x_0\}$

(b) divergent is $(-\infty, x_0) \cup (x_0, \infty)$, also ok: $\{x \in \mathbb{R}\colon x \neq x_0\}$ or $\mathbb{R} \setminus \{x_0\}$.

0.4. Now let $R > 0$ and fill-in the 5 boxes.

Consider the function $y = h(x)$ defined by the power series in (1).

(a) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

$$h'(x) \;=\; \sum_{n=1}^{\infty} \boxed{\;n\,a_n\,(x - x_0)^{n-1}\;} \;. \tag{2}$$

What can you say about the radius of convergence of the power series in (2)? $\boxed{\text{It's the same } R}$.

(b) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\,dx \;=\; \sum_{n=0}^{\infty} \boxed{\;\frac{a_n}{n+1}\,(x - x_0)^{n+1}\;}\Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} \;.$$