0A. Power Series Consider the (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n\,(x-x_0)^n\;, \tag{1}$$

with radius of convergence $R \in [0,\infty]$.
(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)
Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.
If you are having troubles, see the §10.7 handout *Operations on Power Series* at
http://people.math.sc.edu/girardi/m142/handouts/OperationsOnPowerSeries.pdf .

- First let $0 < R < \infty$. The largest set of x's for which we know that the power series in (1) is:

  (a) absolutely convergent is $(x_0 - R, x_0 + R)$, also ok: $\{x \in \mathbb{R}\colon\ |x - x_0| < R\}$

  (b) divergent is $(-\infty, x_0 - R) \cup (x_0 + R, \infty)$, also ok: $\{x \in \mathbb{R}\colon\ |x - x_0| > R\}$.

  What can you say about the convergence of the power series in (1) when $x = x_0 + R$ or $x = x_0 - R$?

  the series can be doing anything, i.e., there are examples showing that it can be absolutely convergent, conditionally convergent or divergent

- Now let $R = \infty$. The largest set of x's for which we know that the power series in (1) is:

  (a) absolutely convergent is $\mathbb{R}$, also ok: $\{x \in \mathbb{R}\colon\ |x - x_0| < R\}$

  (b) divergent is $\emptyset$, also ok: the empty set .

- Now let $R = 0$. The largest set of x's for which we know that the power series in (1) is:

  (a) absolutely convergent is $\{x_0\}$, also ok: $\{x \in \mathbb{R}\colon x = x_0\}$

  (b) divergent is $(-\infty, x_0) \cup (x_0, \infty)$, also ok: $\{x \in \mathbb{R}\colon x \neq x_0\}$ or $\mathbb{R} \setminus \{x_0\}$.

- Now let $R > 0$ and fill-in the 5 boxes.
  Consider the function $y = h(x)$ defined by the power series in (1).

  (a) The function $y = h(x)$ is always differentiable on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

  $$h'(x) \;=\; \sum_{n=1}^{\infty} \boxed{\,n\,a_n\,(x-x_0)^{n-1}\,}\;. \tag{2}$$

  What can you say about the radius of convergence of the power series in (2)? $\boxed{\text{It's the same } R}$.

  (b) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{(x_0 - R, x_0 + R)}$ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

  $$\int_{x=\alpha}^{x=\beta} h(x)\,dx \;=\; \sum_{n=0}^{\infty} \boxed{\frac{a_n}{n+1}\,(x-x_0)^{n+1}}\Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta}\;.$$

0B. Taylor/Maclaurin Polynomials and Series. Fill-in the boxes.
If you are having troubles, see the class handout for §10.8 at
http://people.math.sc.edu/girardi/m142/handouts/16sTaylorPoly.pdf
and the handout for §10.8/9/10 at
http://people.math.sc.edu/girardi/m142/handouts/16sTaylorSeries.pdf .

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. In open form (i.e., with "..." notation and without a $\sum$-sign)

$$P_N(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N}$$

b. In closed form (i.e., with a $\sum$-sign and without "..." notation)

$$P_N(x) = \boxed{\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n}$$

c. In open form (i.e., with "..." notation and without a $\sum$-sign)

$$P_\infty(x) = \boxed{f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots}$$

d. In closed form (i.e., with a $\sum$-sign and without "..." notation)

$$P_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n}$$

e. The formula for c_n is

$$c_n = \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)}} \quad \text{for some } c \text{ between } \boxed{x} \text{ and } \boxed{x_0}\,.$$

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.