0. Fill-in-the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

Sequences

0.1. Practice taking basic limits of sequences. (Important, e.g., for Ratio and Root Tests.) Can you do similar problems?

- $\lim_{n\to\infty} \frac{5n^{17}+6n^2+1}{7n^{18}+9n^3+5} =$ 0
- $\lim_{n\to\infty} \sqrt{\frac{36n^{17}-6n^2-1}{4n^{17}+9n^3+5}} =$ $\sqrt{\frac{36}{4}}$ or 3
- $\lim_{n\to\infty} \frac{-5n^{18}+6n^2+1}{7n^{17}+9n^3+5} =$ DNE or $-\infty$
- $\lim_{n\to\infty} n^{\frac{1}{n}} =$ 1

0.2. Geometric Sequence. Fill in the boxes with with the proper range of $r \in \mathbb{R}$. (Needed for Geometric Series!)

- $\lim_{n\to\infty} r^n = 0$ if and only if r satisfies $|r| < 1$ also ok: $-1 < r < 1$ or $r \in (-1, 1)$.
- $\lim_{n\to\infty} r^n = 1$ if and only if r satisfies $r = 1$.
- the sequence $\{r^n\}_{n=1}^{\infty}$ diverges to ∞ if and only if r satisfies $r > 1$ also ok: $r \in (1, \infty)$.
- the sequence $\{r^n\}_{n=1}^{\infty}$ diverges but does not diverge to ∞ if and only if r satisfies $r \leq -1$ also ok: $r \in (-\infty, -1]$.

0.3. Commonly Occurring Limits of Sequences. Here, $c \in \mathbb{R}$ is a constant. ⟨Thomas Book §10.1, Theorem 5

(1) $\lim_{n\to\infty} \frac{\ln n}{n} =$ 0

(2) $\lim_{n\to\infty} \sqrt[n]{n} =$ 1

(3) $\lim_{n\to\infty} c^{1/n} =$ 1 $(c > 0)$

(4) $\lim_{n\to\infty} c^n =$ 0 $(|c| < 1)$

(5) $\lim_{n\to\infty} \left(1+\frac{c}{n}\right)^n =$ e^c $(c \in \mathbb{R})$

(6) $\lim_{n\to\infty} \frac{x^n}{n!} =$ 0 $(c \in \mathbb{R})$

Series

0.4. For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums, so $s_n = \sum_{k=1}^{n} a_k$. Then the formal series $\sum a_n$:

- converges if and only if [the $\lim_{n\to\infty} s_n$ exists]
- converges to $L \in \mathbb{R}$ if and only if [the $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} s_n = L$]
- diverges if and only if [the $\lim_{n\to\infty} s_n$ does not exist].

Now assume, furthermore, that $a_n \geq 0$ for each n. Then the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums either

- is bounded above (by some finite number), in which case the series $\sum a_n$ [converges]

or

- is not bounded above (by some finite number), in which case the series $\sum a_n$ [diverges to $^{+}\infty$].

0.5. Fix $r \in \mathbb{R}$ with $r \neq 1$. For $N \geq 50$, let $s_N = \sum_{\mathbf{n=50}}^{N} r^n$. (Note the sum starts at 50.) For each $N \geq 50$, the partial sums s_N can be written as: (your answer should NOT contain a "…" nor a "$\sum$" sign)

$$s_N = \boxed{\frac{r^{50} - r^{N+1}}{1-r}} .$$

0.6. Geometric Series. Fill in the boxes with the proper range of $r \in \mathbb{R}$.

- The series $\sum r^n$ converges if and only if r satisfies [$|r| < 1$].

0.7. State the n^{th}**-term test** for an arbitrary series $\sum a_n$.

[If $\lim_{n\to\infty} a_n \neq 0$ (which includes the case that $\lim_{n\to\infty} a_n$ does not exist), then $\sum a_n$ diverges .]

0.8. p**-series**. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if [$p > 1$].

Tests for Positive-Termed Series
(so for $\sum a_n$ where $a_n \geq 0$)

0.9. State the **Integral Test with Remainder Estimate** for a positive-termed series $\sum a_n$.

Let $f\colon [1, \infty) \to \mathbb{R}$ be so that

(1) $a_n = f(n)$ for each $n \in \mathbb{N}$

(2) f is a [positive] function

(3) f is a [continuous] function

(4) f is a [decreasing (nonincreasing is also ok)] function.

Then

- $\sum a_n$ converges if and only if [$\int_{x=1}^{x=\infty} f(x)\,dx$] converges.
- and if $\sum a_n$ converges, then

$$0 \leq \left(\sum_{k=1}^{\infty} a_k\right) - \left(\sum_{k=1}^{N} a_k\right) \leq \boxed{\int_{x=N}^{x=\infty} f(x)\,dx} .$$

0.10. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If [$0 \leq a_n \leq c_n$ (only $a_n \leq c_n$ is also ok b/c given $a_n \geq 0$)] when $n \geq 17$ and [$\sum c_n$ converges], then $\sum a_n$ converges.
- If [$0 \leq d_n \leq a_n$ (need $0 \leq d_n$ part here)] when $n \geq 17$ and [$\sum d_n$ diverges], then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.11. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then [$[\sum b_n$ converges $\iff$ $\sum a_n$ converges $]$]
- If $L = 0$, then [$[\sum b_n$ converges $\implies$ $\sum a_n$ converges $]$] .
- If $L = \infty$, then [$[\sum b_n$ diverges $\implies$ $\sum a_n$ diverges $]$] .

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.12. **Helpful Intuition** Fill in the 3 boxes using: e^x, $\ln x$, x^q. Use each once, and only once.

Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \geq N_q$ then

$$\boxed{\ln x} \leq \boxed{x^q} \leq \boxed{e^x} .$$

Tests for Arbitrary-Termed Series
(so for $\sum a_n$ where $-\infty < a_n < \infty$)

0.13. By definition, for an arbitrary series $\sum a_n$, (fill in these 3 boxes with convergent or divergent).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ is convergent.
- $\sum a_n$ is conditionally convergent if and only if

  $\sum a_n$ is convergent and $\sum |a_n|$ is divergent.
- $\sum a_n$ is divergent if and only if $\sum a_n$ is divergent.

0.14. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \ .$$

- If $\rho < 1$ then $\sum a_n$ converges absolutely.
- If $\rho > 1$ then $\sum a_n$ diverges.
- If $\rho = 1$ then the test is inconclusive.

0.15. State the **Alternating Series Test** (AST) & **Alternating Series Estimation Theorem**.

Let

(1) $u_n \geq 0$ for each $n \in \mathbb{N}$

(2) $\lim_{n\to\infty} u_n = 0$

(3) u_n $>$ (also ok $\geq$) u_{n+1} for each $n \in \mathbb{N}$.

Then

- the series $\sum (-1)^n u_n$ converges. (also ok: $\sum(-1)^{n+1}u_n$ converges or $\sum(-1)^{n-1}u_n$ converges)
- and we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty}(-1)^n u_n$ by the finite sum $\sum_{k=1}^{N}(-1)^k u_k$ and the error (i.e. remainder) satisfies

$$\left| \sum_{k=1}^{\infty} (-1)^k u_k - \sum_{k=1}^{N} (-1)^k u_k \right| \leq u_{N+1} \ .$$