- A power series (p.s.), centered about $x_0 \in \mathbb{R}$ and with coefficients $c_n \in \mathbb{R}$, is a series of the form,

$$\sum_{n=0}^{\infty} c_n \, (x - x_0)^n \overset{\text{note}}{=} c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + c_3 (x - x_0)^3 + \ldots \; . \tag{PS}$$

- If we evaluate a power series (§10.7-10.10) at a (fixed) x, then we get a numerical series (§10-2-10.6).
- A power series converges (abs.) at it's center. since $\sum_{n=0}^{\infty} c_n \, (x - x_0)^n \big|_{x=x_0} = c_0 + 0^1 + 0^2 + \ldots = c_0$.

Thm. A power series centered at x_0 has a radius of convergence $R \in [0, \infty]$ satisfying

$$\sum_{n=0}^{\infty} c_n \, (x - x_0)^n \quad \begin{cases} \text{is absolutely convergent} & \text{when } |x - x_0| < R \\ \text{can do anything} & \text{when } |x - x_0| = R, \text{ i.e. } x = x_0 \pm R, \text{ the endpts} \\ \text{divergent} & \text{when } |x - x_0| > R. \end{cases}$$

Draw a picture

- The interval of convergence of a p.s. is the set of x's for which the p.s. conv. (absolutely or conditionally).
- To find the radius of convergence R, we often use the Ratio or Root Test.

- $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ is a power series repesentation of a function $y = h(x)$ about x_0 (valid in some interval I) provided

$$h(x) \; = \; \lim_{N \to \infty} \sum_{n=0}^{N} c_n (x - x_0)^n \qquad \text{for each } x \in I \; .$$

- Important Example. The geometric series $\sum x^n \overset{\text{note}}{=} \sum 1 (x - 0)^n$ is absolutely convergent when $|x| < 1$ and is divergent when $|x| \geq 1$. So $\sum x^n$ has radius of convergence 1. When $|x| < 1$, we know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ and so $h(x) = \frac{1}{1-x}$ has a power series representation

$$\frac{1}{1-x} \; = \; \sum_{n=0}^{\infty} x^n \quad \text{valid when} \quad x \in (-1, 1) \; . \tag{GS}$$

Setting for rest of the handout.

We find power representations for two (given) functions $y = f(x)$ and $y = g(x)$.

$$f(x) \overset{(*_f)}{:=} \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{valid for} \quad x \in (x_0 - R_f \; , \; x_0 + R_f)$$

$$g(x) \overset{(*_g)}{:=} \sum_{n=0}^{\infty} b_n (x - x_0)^n \quad \text{valid for} \quad x \in (x_0 - R_g \; , \; x_0 + R_g) \; ,$$

where

$0 < R_f \;\; := \;\;$ the radius of convergence of the power series representation $\sum a_n (x - x_0)^n$ of f

$0 < R_g \;\; := \;\;$ the radius of convergence of the power series representation $\sum b_n (x - x_0)^n$ of g .

The equality signs mark as $\overset{\heartsuit}{=}$ denotes that this equality is where the heart of the theorem lies.

Algebraic Operations on Power Series

►. Evaluate a power series at a constant (e.g., plugging in 17 for x) and you get a numerical series.
So the Algebraic Operations that hold for numerical series also hold for power series. So we get Thm1&2:

Theorem 1. For a constant $c \in \mathbb{R}$ and $m \in \mathbb{N}$, we can obtain power series representations for the functions: $y = cf(x)$ and $h(x) = (x - x_0)^m f(x)$.

$$cf(x) \overset{\text{i.e.}}{=} c\left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right] \overset{\heartsuit}{=} \sum_{n=0}^{\infty} c\, a_n (x - x_0)^n$$

$$(x - x_0)^m f(x) \overset{\text{i.e.}}{=} (x - x_0)^m \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right] \overset{\heartsuit}{=} \sum_{n=0}^{\infty} a_n (x - x_0)^m (x - x_0)^n \overset{\text{Ⓐ}}{=} \sum_{n=0}^{\infty} a_n (x - x_0)^{m+n}$$

which are valid for $x \in (x_0 - R_f\ ,\ x_0 + R_f)$.

Theorem 2. We can obtain power series representations for the functions $f \pm g$.

$$f(x) + g(x) \overset{\text{i.e.}}{=} \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right] + \left[\sum_{n=0}^{\infty} b_n (x - x_0)^n\right] \overset{\heartsuit}{=} \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

$$f(x) - g(x) \overset{\text{i.e.}}{=} \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right] - \left[\sum_{n=0}^{\infty} b_n (x - x_0)^n\right] \overset{\heartsuit}{=} \sum_{n=0}^{\infty} (a_n - b_n)(x - x_0)^n$$

which are valid for $x \in (x_0 - R_f\ ,\ x_0 + R_f) \cap (x_0 - R_g\ ,\ x_0 + R_g)$.

Calculus Operations on Power Series

Theorem 3. We can obtain a power series representation for $y = f'(x)$.

$$f'(x) \overset{\text{i.e.}}{=} D_x\left[\sum_{n=\underline{\underline{0}}}^{\infty} a_n(x - x_0)^n\right] \overset{\heartsuit}{=} \sum_{n=\underline{\underline{1}}}^{\infty} D_x(a_n(x - x_0)^n) \overset{\text{©}}{=} \sum_{n=\underline{\underline{1}}}^{\infty} n a_n (x - x_0)^{n-1} \tag{3}$$

which is valid for $x \in (x_0 - R_f\ ,\ x_0 + R_f)$.
Also, the radius of convergence of each power series in (3) is R_f.

Theorem 4. We can also obtain a power series representation for the integral.

$$\begin{aligned}\int f(x)\,dx &\overset{\text{i.e.}}{=} \int \left[\sum_{n=0}^{\infty} a_n(x - x_0)^n\right] dx \overset{\heartsuit}{=} \sum_{n=0}^{\infty}\left[\int a_n(x - x_0)^n\,dx\right] \\ &\overset{\text{©}}{=} C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}\end{aligned} \tag{4}$$

and so if α and β are in $(x_0 - R_f\ ,\ x_0 + R_f)$, then

$$\begin{aligned}\int_{x=\alpha}^{x=\beta} f(x)\,dx &\overset{(4)}{=} \left[\sum_{n=0}^{\infty} \frac{a_n}{n+1}(\beta - x_0)^{n+1}\right] - \left[\sum_{n=0}^{\infty} \frac{a_n}{n+1}(\alpha - x_0)^{n+1}\right] \\ &\overset{\text{Thm 2}}{=} \sum_{n=0}^{\infty} \frac{a_n}{n+1}\left[(\beta - x_0)^{n+1} - (\alpha - x_0)^{n+1}\right].\end{aligned}$$

Also, the radius of convergence of each power series in (4) is R_f.

Warning: we exclude the endpoints $x = x_0 \pm R_f$ since (3) and (4) sometimes don't holds at these endpoints.