Throughout this handout, we are dealing with a function $f\colon [a,b] \to \mathbb{R}$. In recent sections, often our goal was to find the antiderivative F of f, i.e. to find a function $F\colon [a,b] \to \mathbb{R}$ such that

$$\int f(x)\ dx = F(x) + C\ .$$

Now goal is to find the number

$$\int_a^b f(x)\ dx\ .$$

If we can find the antiderivative F, then we know (FTC) that

$$\int_a^b f(x)\ dx = \int_a^b F'(x)\ dx = F(b) - F(a)\ .$$

The trouble is that sometimes F is hard to find ... so we cope ... and numerically approximate $\int_a^b f(x)\ dx$, i.e., we look for some number $L \in R$ so that

$$\int_a^b f(x)\ dx\ \approx\ L\ .$$

We learned (§5.1,5.2) how to numerically approximate $\int_a^b f(x)\ dx$ using Riemann Sums/Rectangles. Now we are going to learn the Trapezoidal (Trap.) Rule, which uses trapezoids instead of rectangles. To use the Trap. Rule, start off just as you would when forming Riemann Rectangles.

- Divide $[a,b]$ into n subintervals $[x_{i-1}, x_i]$, each of length

$$\Delta x = \frac{b-a}{n}$$

- So we get

$$x_i = a + i\,(\Delta x)$$

  for $i = 0, 1, 2, \ldots, n$.

Trapazidal Rule approximation T_n of $\int_a^b f(x)\ dx$ with n-steps is

$$T_n\ =\ \frac{\Delta x}{2}\ \left[1 \cdot f(x_0)\ +\ \left(2 \cdot \sum_{i=1}^{n-1} f(x_i)\right)\ +\ 1 \cdot f(x_n)\right]\ .$$

The terminology n-step refers to the fact that we are dividing the interval $[a,b]$ into n subintervals (of equal length).

Question. $\int_a^b f(x)\ dx \approx T_n$ but how good is this approximatation $\approx$?

Answer. The Trapezoidal Rule Error Theorem tells us!

The useful fact/theorem. Let f'' be continuous on $[a,b]$. Then there exists $c \in [a,b]$ so that

$$\left|T_n - \int_a^b f(x)\ dx\right|\ =\ \left(\frac{(b-a)^3}{12n^2}\right)\ \cdot\ \left[\ |f''(c)|\ \right]\ .$$

Unfortunately, we do not know where in the interval $[a,b]$ the point c is but at least we can say:

Trapezoidal Rule Error Estimate (TREE)

Let f'' be continuous on $[a,b]$. Then

$$\left|T_n - \int_a^b f(x)\ dx\right|\ \leq\ \left(\frac{(b-a)^3}{12n^2}\right)\ \cdot\ \left[\max_{a \leq x \leq b} |f''(x)\ |\right] \qquad \text{(TREE)}$$

where $\left[\max_{a \leq x \leq b} |f''(x)|\right]$ is the maximum value (of y) the function $y = |f''(x)|$ can be when $x \in [a,b]$.

Do not use a calculator on Problems 1-4. Your answers should be numbers. However, you do not have to perform grade school level arithmetic. E.g., either expression below would be acceptable.

$$\frac{3}{10}\left[\frac{13}{17}+(2)\sqrt{\frac{14}{15}}+\sqrt{\frac{16}{17}}\right] \qquad \text{or} \qquad \frac{(17)(18)}{(19)(20)(21)(22)}\ .$$

Let

$$f(x)=\sqrt{x} \qquad \text{and} \qquad [a,b]=[4,5]\ .$$

1. Find the Trapezidal Rule approximation T_3 of $\int_a^b f(x)\,dx$ with 3 steps.

ANSWER: $T_3 =$

$\Delta x =$			Table for T_3.
i	x_i	weight (1 or 2)	$f(x_i)$
0			
1			
2			
3			

2. Find a good upper bound for $\max_{a\le x\le b}|f''(x)|$.

ANSWER: $\max\limits_{a\le x\le b}|f''(x)| \le$

3. The Trapezoidal Rule Error Estimate gives that $\left|T_3-\int_a^b f(x)\,dx\right| \le$ $\boxed{}$.

4. Now take an arbitary integer $n\in\mathbb{N}$. Consider the Trap. rule approximation T_n of $\int_a^b f(x)\,dx$ with n steps.

The Trapezoidal Rule Error Estimate gives that $\left|T_n-\int_a^b f(x)\,dx\right| \le$ $\boxed{}$.

5. Find the smallest integer $n\in\mathbb{N}$ so that the Trapezoidal Rule Error Estimate guarantees that $\left|T_n-\int_a^b f(x)\,dx\right| \le 10^{-4}$. ANSWER: The integer $n =$ $\boxed{}$.

Recall: $1\mathrm{E}^{-4} = 10^{-4} = \frac{1}{(10)^4} = \frac{1}{10000} = 0.0001$