TODAY'S GOAL
THE p-SERIES

Determine the behavior of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. We will show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent to } {}^{+}\infty & \text{if } p \leq 1 \ . \end{cases}$$

When $p = 1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ is also called the harmonic series.

Note that the p-series looks like

$$p\text{-series}: \quad \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$
$$= 1 + \left(\frac{1}{2}\right)^p + \left(\frac{1}{3}\right)^p + \left(\frac{1}{4}\right)^p + \ldots .$$

Warning: do not confuse a p-series $\sum_n \frac{1}{n^p}$ with a geometric seris $\sum_n r^n$;

$$\text{geometric series}: \quad \sum_{n=1}^{\infty} r^n = r^1 + r^2 + r^3 + r^4 + \ldots .$$

RECALL

Recall that in the section on Improper Integrals, we showed that

$$\int_{x=1}^{x=\infty} \frac{1}{x^p}\, dx \quad \text{is} \quad \begin{cases} \text{convergent to } \frac{1}{p-1} & \text{if } p > 1 \\ \text{divergent to } {}^{+}\infty & \text{if } p \leq 1 \ . \end{cases}$$

INTEGRAL TEST

Let's say we are given a series $\sum a_n$ and we can find a function $f\colon [\,\underline{\underline{1}}\,,\ \infty) \to \mathbb{R}$ satisfying

(1) $a_n = f(n)$ for each $n \in \mathbb{N}$ with $n \geq \underline{\underline{1}}$ (this is usually accomplished by design)

(2) $y = f(x)$ is positive on $[\,\underline{\underline{1}}\,, \infty)$ (so $\sum a_n$ needs to be a positive term series)

(3) $y = f(x)$ is continuous on $[\,\underline{\underline{1}}\,, \infty)$

(4) $y = f(x)$ is decreasing on $[\,\underline{\underline{1}}\,, \infty)$ (can confirm this by showing $f'(x) \leq 0$).

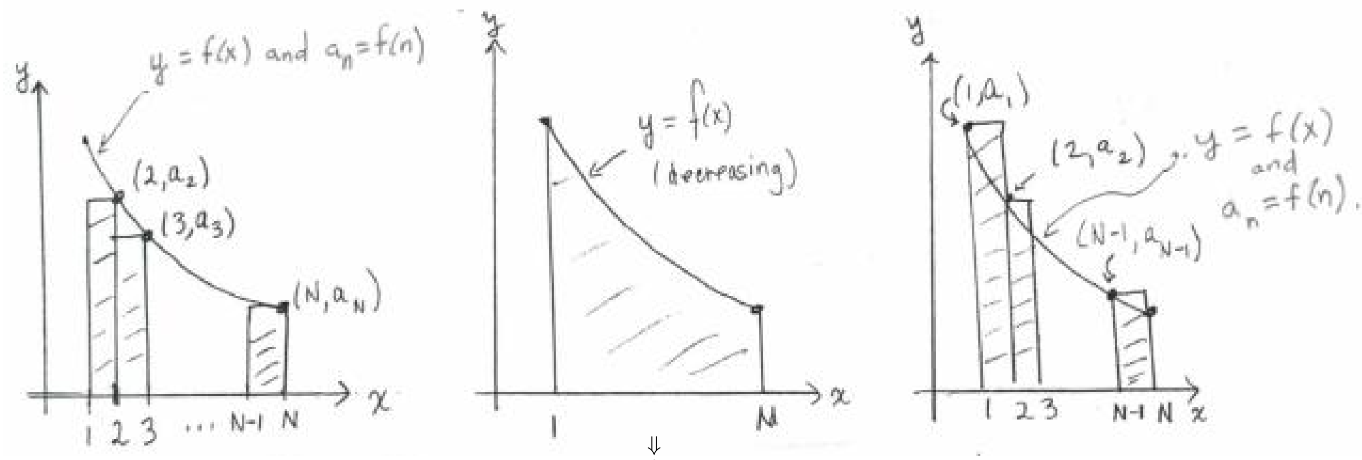
Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_{x=\underline{\underline{1}}}^{x=\infty} f(x)\, dx$ either:

(a) both converge (to finite numbers, although most likely different numbers)

(b) both diverge (to ∞) .

WHY THE INTEGRAL TEST IS TRUE

Let's say we are given a series $\sum a_n$ and find a function $f\colon [1\ ,\ \infty) \to \mathbb{R}$ satisfying (1)–(4). Then
the sequence $\{\sum_{k=1}^{n} a_k\}_{n\in\mathbb{N}}$ and the sequence $\{\int_1^n f(x)\,dx\}_{n\in\mathbb{N}}$
are both increasing sequences and so each sequence has the choice of
either [converging to some finite number] or [diverging to ∞].
Next compare the terms of these two sequences:



$$\overbrace{a_2 + a_3 + \ldots + a_N}^{\Downarrow} \quad \leq \quad \overbrace{\int_{x=1}^{x=N} f(x)\,dx}^{\Downarrow} \quad \leq \quad \overbrace{a_1 + a_2 + \ldots + a_{N-1}}^{\Downarrow} \ .$$

Now take the limit as $N \to \infty$ to see that

$$\sum_{k=2}^{\infty} a_k \quad \overset{\text{(A)}}{\leq} \quad \int_{x=1}^{x=\infty} f(x)\,dx \quad \overset{\text{(B)}}{\leq} \quad \sum_{k=1}^{\infty} a_k \ . \qquad (*_1)$$

The integral test now follows from $(*_1)$.

What if we changed our interval $[1, \infty)$ to $[17, \infty)$? Then $(*_1)$ would become

$$\sum_{k=17+1}^{\infty} a_k \quad \overset{(\text{A}_{17})}{\leq} \quad \int_{x=17}^{x=\infty} f(x)\,dx \quad \overset{(\text{B}_{17})}{\leq} \quad \sum_{k=17}^{\infty} a_k \ . \qquad (*_{17})$$

Observation 1. The statement of the Integral Test remains true if we replace each 1 with 17, or any other integer. This is useful if, e.g., you can get (1)–(3) to hold but only have $y = f(x)$ decreasing on $[17, \infty)$.

Observation 2. . The Integral Test Remainder Estimate.
Let's say that we have shown that $\sum a_n$ converges by using the integral test with the function $y = f(x)$, which satisfies that above conditions (1) - (4). Then we can approximate the infinite sum $S := \sum_{n=1}^{\infty} a_n$ by the computable finite sum $s_n := \sum_{k=1}^{n} a_k$. Indeed, define S, s_n, and R_n by

$$S := \sum_{n=1}^{\infty} a_n \quad \text{and} \quad s_n := \sum_{k=1}^{n} a_k \quad \text{and} \quad S \ = \ s_n \ + \ R_n \ .$$

Then $S \approx s_n$ within an error of $|R_n|$ and we can bound R_n by

$$0 \overset{\text{by (2)}}{\leq} \int_{x=n+1}^{x=\infty} f(x)\,dx \overset{\text{by }(\text{B}_{17})}{\leq} \boxed{R_n \overset{\text{note}}{=} \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k \overset{\text{note}}{=} \sum_{k=n+1}^{\infty} a_k} \overset{\text{by }(\text{A}_{17})}{\leq} \int_{x=n}^{x=\infty} f(x)\,dx \ .$$