The Fundamental Theorem of Calc. (FTC) gives that if $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then

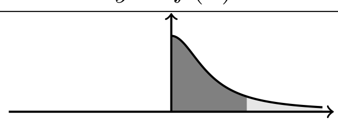
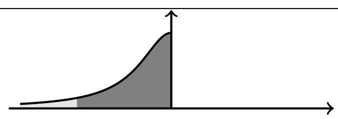
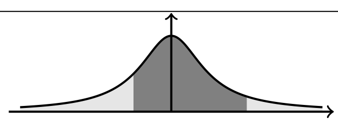
$$\int_a^b f(x)\ dx \ = \ F(b) - F(a)$$

where F is any antiderivative of f, i.e., a function such that $F' = f$. Note that to apply the FTC, one needs f to be continuous on the whole interval $[a,b]$, i.e., we need the following four conditions:

(1) f is defined on the (entire) closed bounded interval $[a,b]$.
(2) If $a < c < b$ (in other words, if $c \in (a,b)$), then (for $y = f(x)$ to be cont. at $x = c$) $\lim_{x\to c} f(x) = f(c)$.
(3) The limit of $y = f(x)$ as x approaches a from the right is $f(a)$, i.e., $\lim_{x\to a^+} f(x) = f(a)$.
(4) The limit of $y = f(x)$ as x approaches b from the left is $f(b)$, i.e., $\lim_{x\to b^-} f(x) = f(b)$.

To apply the FTC, we need to be in the situation where f is continuous on a closed bounded interval. Next we explore when this is *almost* the situtation.

Improper Integrals

	draw a picture of $y = f(x)$	**If f is continuous on**	**then we define**	**by**
1.		$[0,\infty)$	$\int_0^\infty f(x)\ dx$	$\lim_{t\to\infty} \int_0^t f(x)\ dx$
2.		$(-\infty,0]$	$\int_{-\infty}^0 f(x)\ dx$	$\lim_{t\to-\infty} \int_t^0 f(x)\ dx$
3.		$(-\infty,\infty)$	$\int_{-\infty}^\infty f(x)\ dx$	$\left[\lim_{t\to-\infty} \int_t^0 f(x)\ dx\right] + \left[\lim_{s\to\infty} \int_0^s f(x)\ dx\right]$
Above in 1.–3., we can replace 0 by any real number (e.g. 17).				
4.		$(a,b]$	$\int_a^b f(x)\ dx$	$\lim_{t\to a^+} \int_t^b f(x)\ dx$
5.		$[a,b)$	$\int_a^b f(x)\ dx$	$\lim_{t\to b^-} \int_a^t f(x)\ dx$
6.		$[a,c)\cup(c,b]$	$\int_a^b f(x)\ dx$	$\left[\lim_{t\to c^-} \int_a^t f(x)\ dx\right] + \left[\lim_{s\to c^+} \int_s^b f(x)\ dx\right]$

- The improper integral *converges* $\Leftrightarrow$ **each** of the limits involves converges to a finite number.
- The improper integral *diverges* if and only if the integral does not converge.

Important example from this section!

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \text{is} \qquad \begin{cases} \text{divergent (to } \infty) & \text{when } p \le 1. \\ \text{convergent (to the finite number } \frac{1}{p-1}) & \text{when } p > 1\ . \end{cases}$$

If $p = 1$:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\left[\ln(x)\Big|_{x=1}^{x=t}\right] = \lim_{t\to\infty}\left[\ln(t) - \ln(1)\right] = \lim_{t\to\infty}\ln(t) = \infty.$$

If $p \neq 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\Big|_{x=1}^{x=t}\right] = \left(\frac{1}{1-p}\right)\lim_{t\to\infty}\left[x^{1-p}\Big|_{x=1}^{x=t}\right] = \left(\frac{1}{1-p}\right)\lim_{t\to\infty}\left[t^{1-p} - 1\right].$$

And so if $p < 1$ (i.e, equivalently, $0 < 1 - p$), then

$$\int_1^\infty \frac{1}{x^p}\,dx = \left(\frac{1}{1-p}\right)\lim_{t\to\infty}\left[t^{1-p} - 1\right] = \infty$$

while if $p > 1$ (or equivalently $p - 1 > 0$), then

$$\int_1^\infty \frac{1}{x^p}\,dx = \left(\frac{1}{1-p}\right)\lim_{t\to\infty}\left[t^{1-p} - 1\right] = \left(\frac{1}{1-p}\right)\lim_{t\to\infty}\left[\frac{1}{t^{\,p-1}} - 1\right] = \left(\frac{1}{1-p}\right)[0-1] = \frac{-1}{1-p} = \frac{1}{p-1}.$$

How can an improper integral diverge?

There are many different reasons why an improper integral is divergent. Below are just 2 reasons.

(1) An improper integral <u>diverges to ∞</u> if and only if
   (a) at least one of the involved limits diverges to ∞
   (b) <u>and</u> each of the involved limits EITHER diverges to ∞ OR converges to a finite number.

(2) An improper integral <u>diverges to $-\infty$</u> if and only if
   (a) at least one of the involved limits diverges to $-\infty$
   (b) <u>&</u> each of the involved limits EITHER diverges to $-\infty$ OR converges to a finite number.

(3) Can you think of a divergent improper integral that does not diverge to ∞ and also does not diverge to $-\infty$?

Helpful

We will use the following standard convention (where $L \in \mathbb{R}$) for divergent improper integrals.

$$\begin{array}{lcl} (\infty) + (\infty) = \infty & \qquad & (-\infty) + (-\infty) = -\infty \\ L + (\infty) = \infty & & L + (-\infty) = -\infty \\ (\infty) + L = \infty & & (-\infty) + L = -\infty \end{array}$$

However, the following expressions are undefined (and so *do not exist*, a.k.a. DNE)

$$(\infty) + (-\infty) \qquad\qquad (-\infty) + (\infty)\ .$$

(This is inline with what you learned when studying L'Hopital's Rule: $\infty - \infty$ is undefined.)

Goal

You are given a continuous, nonnegative function

$$f\colon [a,\infty) \to [0,\infty)$$

and you want to determine whether $\int_a^\infty f(x)\, dx$ is convergent or divergent.

Key Idea

Since $f(x) \geq 0$, we know $\int_a^\infty f(x)\, dx$ must either converges (to a finite number) or diverges to ∞. One way to detemine this is to COMPARE f to a continuous, nonnegative function $g\colon [a,\infty) \to [0,\infty)$ where you KNOW whether $\int_a^\infty g(x)\, dx$ is convergent or divergent.

Direct Comparison Test (DCT)

1. If $0 \leq f(x) \leq g(x)$ for each $x \in [a,\infty)$ and $\int_0^\infty g(x)\, dx$ converges, then $\int_0^\infty f(x)\, dx$ converges.
2. If $0 \leq g(x) \leq f(x)$ for each $x \in [a,\infty)$ and $\int_0^\infty g(x)\, dx$ diverges, then $\int_0^\infty f(x)\, dx$ diverges.

The DCT holds (also serves as a way to remember the DCT) since

$$0 \leq f \leq g \qquad \Rightarrow \qquad 0 \leq \int_a^\infty f(x)\, dx \leq \int_a^\infty g(x)\, dx$$

$$0 \leq g \leq f \qquad \Rightarrow \qquad 0 \leq \int_a^\infty g(x)\, dx \leq \int_a^\infty f(x)\, dx$$

Sing the song:

♪♪ Bound above by a convergent, below by a divergent. ♪♪

Limit Comparison Test (LCT)

Let $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ and $\boxed{0 < L < \infty}$.
Then we now that for x sufficiently large (i.e. big enough)

$$\left(\frac{L}{2}\right) g(x) \;\leq\; f(x) \;\leq\; (2L)\; g(x) \;.$$

So

$$\int_0^\infty f(x)\, dx \text{ converges} \qquad \Longleftrightarrow \qquad \int_0^\infty g(x)\, dx \text{ converges}$$

which is the same as saying

$$\int_0^\infty f(x)\, dx \text{ diverges} \qquad \Longleftrightarrow \qquad \int_0^\infty g(x)\, dx \text{ diverges}$$

i.e., $\int_0^\infty f(x)\, dx$ and $\int_0^\infty g(x)\, dx$ will "do the same thing."

1. Determine whether the following (improper) integral is convergent or divergent.

$$\int_2^{\infty} \frac{dx}{1+e^x}$$

Comment. The indefinite integral $\int \frac{dx}{1+e^x}$ is do-able but it takes some work (first do a substitution $u = e^x$ and then do partial fractions) to show that $\int \frac{dx}{1+e^x} = x - \ln(1+e^x) + C$. Let's try to reduce our work by using a comparison test.

Thinking Land. Let

$$f(x) = \frac{1}{1+e^x} \qquad \text{where} \qquad f\colon [2,\infty) \to [0,\infty) \ .$$

We want to compare f to a continuous nonnegative $g\colon [2,\infty) \to [0,\infty)$, where we can EASILY figure out what $\int_2^\infty g(x)\, dx$ does. When x is big (think of as close to ∞)

$$\frac{1}{1+e^x} \approx \frac{1}{e^x}$$

and $\int \frac{1}{e^x}\, dx$ is alot easier to integrate than $\int \frac{1}{1+e^x}\, dx$. So we will try comparing f to

$$g(x) := \frac{1}{e^x} \qquad \text{where} \qquad g\colon [2,\infty) \to [0,\infty) \ .$$

Compute

$$\int_2^t \frac{dx}{e^x} = \int_2^t e^{-x}\, dx = -e^{-x} \Big|_{x=2}^{x=t} = \frac{1}{e^x} \Big|_{x=t}^{x=2} = \frac{1}{e^2} - \frac{1}{e^t} \xrightarrow{t\to\infty} \frac{1}{e^2} - 0 = e^{-2} \ .$$

So $\int_2^\infty \frac{dx}{e^x}$ converges (in fact, $\int_2^\infty \frac{dx}{e^x} = e^{-2}$).

Direct Comparison Test.

If $x \in [2,\infty)$[1], then

$$e^x \le 1 + e^x \qquad \text{and so} \qquad f(x) = \frac{1}{1+e^x} \le \frac{1}{e^x} = g(x) \ . \tag{1}$$

So by the direct comparison test (we just ♪bound above by a convergent ♪), $\boxed{\int_2^\infty \frac{dx}{1+e^x} \text{ converges}}$.

Note that the direct comparison test (DCT) does not tell us what number $\int_2^\infty \frac{dx}{1+e^x}$ converges to; all the DCT tells us is that $\int_2^\infty \frac{dx}{1+e^x}$ converges to some number and that $\int_2^\infty \frac{dx}{1+e^x} \le \int_2^\infty \frac{dx}{e^x} = e^{-2}$.

Limit Comparison Test.

Compute

$$L := \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^x}{1+e^x} \underset{\text{L'H}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1 \ .$$

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is strictly between 0 & ∞ and furthermore $\int_2^\infty g(x)\, dx$ converges, the Limit Comparison Test (LCT) tells us that $\boxed{\int_2^\infty f(x)\, dx \text{ converges}}$.

As with the DCT, the LCT does not tell us to what precise number $\int_2^\infty f(x)\, dx$ converges.

2. Determine whether the following (improper) integral is convergent or divergent.

$$\int_2^{\infty} \frac{dx}{-1+e^x} \tag{2}$$

[1]Recall $x \in [2,\infty)$ reads "x is an element of the set $[2,\infty)$" and so $x \in [2,\infty)$ is just saying $x \ge 2$.

3. Determine whether the following (improper) integral is convergent or divergent.

$$\int_{x=1}^{\infty} \frac{x+1}{\sqrt{x^4+x}}\,dx \tag{3}$$

converges or diverges.

The integral in (3) is an improper integral with

$$\int_{x=1}^{x=\infty} \frac{x+1}{\sqrt{x^4+x}}\,dx \;=\; \lim_{b\to\infty}\int_1^b \frac{x+1}{\sqrt{x^4+x}}\,dx.$$

Thinking land. The function $f(x) = \frac{x+1}{\sqrt{x^4+x}}$ is hard to integrate. So we look for a function g such that $f(x) \approx g(x)$ if x is really big (after all, we are integrating out to infinity) AND g is easy to integrate. Well, when x is really big,

$$f(x) = \frac{x+1}{\sqrt{x^4+x}} \approx \frac{x}{\sqrt{x^4}} = \frac{x}{x^2} = \frac{1}{x} := g(x)\ ,$$

i.e, we let

$$g(x) = \frac{1}{x}\ .$$

For the LCT, we next let $L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$ and we need to check that

(1) the limit L actually exists

(2) $0 < L < \infty$.

So let's do some algebra:

$$\frac{f(x)}{g(x)} = \frac{\frac{x+1}{\sqrt{x^4+x}}}{\frac{1}{x}} = \left(\frac{x+1}{\sqrt{x^4+x}}\right)\left(\frac{x}{1}\right) = \frac{x^2+x}{\sqrt{x^4+x}}\ . \tag{4}$$

The limit

$$\lim_{x\to\infty} \frac{x^2+x}{\sqrt{x^4+x}}$$

is do-able by several applications of L'Hopital's rule and the product rule. Let's do some more (clever but simple) algebra to (4) as so to be able to easily compute the limit. (The below step of dividing the numerator and denomerator by x^2 may seem unmotivated but after you see how nice it makes life, ask yourself *why does this work and how/when can I use it again*)

$$\frac{f(x)}{g(x)} \overset{\text{from (4)}}{=} \frac{x^2+x}{\sqrt{x^4+x}} = \frac{\frac{x^2+x}{x^2}}{\frac{\sqrt{x^4+x}}{x^2}} = \frac{\frac{x^2}{x^2}+\frac{x}{x^2}}{\sqrt{\frac{x^4+x}{x^4}}} = \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^3}}} \xrightarrow{x\to\infty} \frac{1+0}{\sqrt{1+0}} = 1 := L. \tag{5}$$

Since $\lim_{x\to\infty} \frac{\frac{x+1}{\sqrt{x^4+x}}}{\frac{1}{x}} = 1$ and $0 < 1 < \infty$, the LCT says $\int_1^\infty \frac{x+1}{\sqrt{x^4+x}}\,dx$ and $\int_1^\infty \frac{1}{x}\,dx$ *do the same thing*, i.e., either both converge (to finite, but maybe different, numbers) or both diverge (to ∞).

Since $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \,\big|_{x=1}^{x=b} = \lim_{b\to\infty}[\ln b - \ln 1] = \infty - 0 = \infty$,

$\int_1^\infty \frac{1}{x}\,dx$ diverges to ∞.

So, by LCT, $\int_1^\infty \frac{x+1}{\sqrt{x^4+x}}\,dx$ also diverges to ∞.