Variant of # 51. Using the Direct Comparsion Test (DCT), determine whether the integral

$$\int_{x=1}^{\infty} \frac{x+1}{\sqrt{x^4+x}}\, dx \tag{1}$$

converges or diverges.

The integral in (1) is an improper integral with

$$\int_{x=1}^{\infty} \frac{x+1}{\sqrt{x^4+x}}\, dx \;=\; \lim_{b\to\infty} \int_{1}^{b} \frac{x+1}{\sqrt{x^4+x}}\, dx.$$

Thinking land. The function $f(x) = \frac{x+1}{\sqrt{x^4+x}}$ is hard to integrate. So we look for a function g such that $f(x) \approx g(x)$ if x is really big (after all, we are integrating out to infinity) AND g is easy to integrate. Well, when x is really big,

$$f(x) = \frac{x+1}{\sqrt{x^4+x}} \approx \frac{x}{\sqrt{x^4}} = \frac{x}{x^2} = \frac{1}{x} := g(x) \;\; ,$$

i.e, we let

$$g(x) = \frac{1}{x} \; .$$

For the LCT, we next let $L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$ and we need to check that

(1) the limit L actually exists

(2) $0 < L < \infty$.

So let's do some algebra:

$$\frac{f(x)}{g(x)} = \frac{\frac{x+1}{\sqrt{x^4+x}}}{\frac{1}{x}} = \left(\frac{x+1}{\sqrt{x^4+x}}\right)\left(\frac{x}{1}\right) = \frac{x^2+x}{\sqrt{x^4+x}} \; . \tag{2}$$

The limit

$$\lim_{x\to\infty} \frac{x^2+x}{\sqrt{x^4+x}}$$

is do-able by several applications of L'Hopital's rule and the product rule. Let's do some more (clever but simple) algebra to (2) as so to be able to easily compute the limit. (The below step of dividing the numerator and denomerator by n^2 may seem unmotivated but after you see how nice it makes life, ask yourself *why does this work and how/when can I use it again*)

$$\frac{f(x)}{g(x)} \overset{\text{from (2)}}{=} \frac{x^2+x}{\sqrt{x^4+x}} = \frac{\frac{x^2+x}{x^2}}{\frac{\sqrt{x^4+x}}{x^2}} = \frac{\frac{x^2}{x^2}+\frac{x}{x^2}}{\sqrt{\frac{x^4+x}{x^4}}} = \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^3}}} \; . \tag{3}$$

Now it is easy to take $\lim_{x\to\infty}$ of the right-hand side of (3).

You should be able to finish the problem from here.