Goal

You are given a continuous, nonnegative function

$$f\colon [a,\infty) \to [0,\infty)$$

and you want to determine whether $\int_a^\infty f(x)\ dx$ is convergent or divergent.

Key Idea

Since $f(x) \geq 0$, we know $\int_a^\infty f(x)\ dx$ must either converges (to a finite number) or diverges to ∞. One way to detemine this is to COMPARE f to a continuous, nonnegative function $g\colon [a,\infty) \to [0,\infty)$ where you KNOW whether $\int_a^\infty g(x)\ dx$ is convergent or divergent.

Direct Comparison Test (DCT)

1. If $0 \leq f(x) \leq g(x)$ for each $x \in [a,\infty)$ and $\int_0^\infty g(x)\ dx$ converges, then $\int_0^\infty f(x)\ dx$ converges.
2. If $0 \leq g(x) \leq f(x)$ for each $x \in [a,\infty)$ and $\int_0^\infty g(x)\ dx$ diverges, then $\int_0^\infty f(x)\ dx$ diverges.

The DCT holds (also serves as a way to remember the DCT) since

$$0 \leq f \leq g \qquad \Rightarrow \qquad 0 \leq \int_a^\infty f(x)\ dx \leq \int_a^\infty g(x)\ dx$$

$$0 \leq g \leq f \qquad \Rightarrow \qquad 0 \leq \int_a^\infty g(x)\ dx \leq \int_a^\infty f(x)\ dx$$

Sing the song:

♪♪ Bound above by a convergent, below by a divergent. ♪♪

Limit Comparison Test (LCT)

Let $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ and $\boxed{0 < L < \infty}$.
Then we now that for x sufficiently large (i.e. big enough)

$$\left(\frac{L}{2}\right)\ g(x)\ \leq\ f(x)\ \leq\ (2L)\ g(x)\ .$$

So

$$\int_0^\infty f(x)\ dx \text{ converges} \qquad \Longleftrightarrow \qquad \int_0^\infty g(x)\ dx \text{ converges}$$

which is the same as saying

$$\int_0^\infty f(x)\ dx \text{ diverges} \qquad \Longleftrightarrow \qquad \int_0^\infty g(x)\ dx \text{ diverges}$$

i.e., $\int_0^\infty f(x)\ dx$ and $\int_0^\infty g(x)\ dx$ will "do the same thing."

1. Determine whether the following (improper) integral is convergent or divergent.

$$\int_2^\infty \frac{dx}{1+e^x}$$

Comment. The indefinite integral $\int \frac{dx}{1+e^x}$ is do-able but it takes some work (first do a substitution $u = e^x$ and then do partial fractions) to show that $\int \frac{dx}{1+e^x} = x - \ln(1+e^x) + C$. Let's try to reduce our work by using a comparison test.

Thinking Land. Let

$$f(x) = \frac{1}{1+e^x} \qquad \text{where} \qquad f\colon [2,\infty) \to [0,\infty)\ .$$

We want to compare f to a continuous nonnegative $g\colon [2,\infty) \to [0,\infty)$, where we can EASILY figure out what $\int_2^\infty g(x)\, dx$ does. When x is big (think of as close to ∞)

$$\frac{1}{1+e^x} \approx \frac{1}{e^x}$$

and $\int \frac{1}{e^x}\, dx$ is alot easier to integrate than $\int \frac{1}{1+e^x}\, dx$. So we will try comparing f to

$$g(x) := \frac{1}{e^x} \qquad \text{where} \qquad g\colon [2,\infty) \to [0,\infty)\ .$$

Compute

$$\int_2^t \frac{dx}{e^x} = \int_2^t e^{-x}\, dx = -e^{-x}\ \Big|_{x=2}^{x=t} = \frac{1}{e^x}\ \Big|_{x=t}^{x=2} = \frac{1}{e^2} - \frac{1}{e^t} \xrightarrow{t\to\infty} \frac{1}{e^2} - 0 = e^{-2}\ .$$

So $\int_2^\infty \frac{dx}{e^x}$ converges (in fact, $\int_2^\infty \frac{dx}{e^x} = e^{-2}$).

Direct Comparison Test.

If $x \in [2,\infty)$[1], then

$$e^x \le 1 + e^x \qquad \text{and so} \qquad f(x) = \frac{1}{1+e^x} \le \frac{1}{e^x} = g(x)\ . \tag{1}$$

So by the direct comparison test (we just ♪bound above by a convergent ♪), $\boxed{\int_2^\infty \frac{dx}{1+e^x} \text{ converges}}$.

Note that the direct comparison test (DCT) does not tell us what number $\int_2^\infty \frac{dx}{1+e^x}$ converges to; all the DCT tells us is that $\int_2^\infty \frac{dx}{1+e^x}$ converges to some number and that $\int_2^\infty \frac{dx}{1+e^x} \le e^{-2}$.

Limit Comparison Test.

Compute

$$L := \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^x}{1+e^x} \underset{\text{L'H}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1\ .$$

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is strictly between 0 & ∞ and furthermore $\int_2^\infty g(x)\, dx$ converges, the Limit Comparison Test (LCT) tells us that $\boxed{\int_2^\infty f(x)\, dx \text{ converges}}$.

As with the DCT, the LCT does not tell us to what precise number $\int_2^\infty f(x)\, dx$ converges.

2. Determine whether the following (improper) integral is convergent or divergent.

$$\int_2^\infty \frac{dx}{-1+e^x}$$

[1] Recall $x \in [2,\infty)$ reads "x is an element of the set $[2,\infty)$" and so $x \in [2,\infty)$ is just saying $x \ge 2$.