0. Fill-in-the boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

0a. Sequences Fill in the boxes with with the proper range of $r \in \mathbb{R}$. (Afterall, this is needed for Geometric Series!)

- $\lim_{n\to\infty} r^n = 0$ if and only if r satisfies $|r| < 1$ also ok: $-1 < r < 1$ or $r \in (-1, 1)$.
- $\lim_{n\to\infty} r^n = 1$ if and only if r satisfies $r = 1$.
- the sequence $\{r^n\}_{n=1}^{\infty}$ diverges to ∞ if and only if r satisfies $r > 1$ also ok: $r \in (1, \infty)$.
- the sequence $\{r^n\}_{n=1}^{\infty}$ diverges but does not diverge to ∞ if and only if r satisfies $r \leq -1$ also ok: $r \in (-\infty, -1]$.

0b. Fix $r \in \mathbb{R}$ with $r \neq 1$. For $N \geq 1700$, let $s_N = \sum_{\mathbf{n=1700}}^{N} r^n$. Note the sum starts at 1700. Then s_N can be written as:

$$s_N = \frac{r^{1700} - r^{N+1}}{1 - r} .$$

for all $N \geq 1700$. Your answer should NOT contain a "…" nor a "$\sum$" sign.

0c. State the n^{th}**-term test** for an arbitrary series $\sum a_n$.

If $\lim_{n\to\infty} a_n \neq 0$ (which includes the case that $\lim_{n\to\infty} a_n$ does not exist), then $\sum a_n$ diverges .

0d. Geometric Series. Fill in the boxes with the proper range of $r \in \mathbb{R}$. (Hint: look at the previous questions.)

- The series $\sum r^n$ converges if and only if r satisfies $|r| < 1$.
- The series $\sum r^n$ diverges if and only if r satisfies $|r| \geq 1$.

0e. p**-series**. Fill in the boxes with the proper range of $p \in \mathbb{R}$.

- The series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.
- The series $\sum \frac{1}{n^p}$ diverges if and only if $p \leq 1$.

Tests for Positive-Termed Series
(so for $\sum a_n$ where $a_n \geq 0$)

0f. State the **Integral Test** for a positive-termed series $\sum a_n$.
Let $f\colon [1, \infty) \to \mathbb{R}$ be so that

- $a_n = f\left(n \right)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a continuous function
- f is a decreasing (nonincreasing is also ok) function.

Then $\sum a_n$ converges if and only if $\int_{x=1}^{x=\infty} f(x)\,dx$ converges.

0g. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.
Let $N_0 \in \mathbb{N}$.

- If $0 \leq a_n \leq c_n$ when $n \geq N_0$ and $\sum c_n$ converges , then $\sum a_n$ converges.
- If $0 \leq d_n \leq a_n$ when $n \geq N_0$ and $\sum d_n$ diverges , then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0h. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.
Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then [$\sum b_n$ converges $\iff$ $\sum a_n$ converges] .
- If $L = 0$, then [$\sum b_n$ converges $\implies$ $\sum a_n$ converges] .
- If $L = \infty$, then [$\sum b_n$ diverges $\implies$ $\sum a_n$ diverges] .

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

Helpful Intuition

Claim 1: If $x > 0$, then

$$\ln x \leq x^1 \leq e^x \ .$$

To see this , consider the function $g(x) = e^x - x$. Then $g(0) = 1$ and $g'(x) = e^x > 0$ for $x > 0$. So $g(x) > 0$ for $x > 0$. Recall that the graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ over the line $y = x$.

Claim 2: Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \geq N_q$ then

$$\boxed{\ln x \leq x^q \leq e^x} \ .$$

To see Claim 2, use L'Hôpital's rule to show that

$$\lim_{x\to\infty} \frac{\log_e x}{x^q} \ = \ 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{x^q}{e^x} \ = \ 0 \ . \tag{$*$}$$

Recall that $\log_e x = \ln x$. Recall that for any base $b > 0$ with $b \neq 1$

$$\log_b x = \frac{\log_e x}{\log_e b} \qquad \text{and} \qquad D_x \log_b x = \frac{1}{x \ln b} \qquad \text{and} \qquad D_x b^x = b^x \ln b$$

and $\lim_{x\to\infty} b^x = \infty$ if and only if $b > 1$. And so $(*)$ holds if one replaces e with any base $b > 1$.

Claim 3: Consider a positive power $q > 0$ along with a base $b > 1$. There is (some big #) $N_{q,b} > 0$ so that if $x \geq N_{q,b}$ then

$$\log_b x \leq x^q \leq b^x$$

Moral: To figure out what is happening to a series involving $\log_b n$ or b^n, keep in mind that as $n \to \infty$

- $\log_b n$ grows *super slow* compared to n^q
- b^n grows *super fast* compared to n^q

for any positive power $q > 0$ and base $b > 1$.