0. Fill-in-the boxes. All series \( \sum \) are understood to be \( \sum_{n=1}^{\infty} \), unless otherwise indicated.

**0a. Sequences** Fill in the boxes with with the proper range of \( r \in \mathbb{R} \). (Afterall, this is needed for Geometric Series!)

- \( \lim_{n \to \infty} r^n = 0 \) if and only if \( r \) satisfies \( \ldots \).
- \( \lim_{n \to \infty} r^n = 1 \) if and only if \( r \) satisfies \( \ldots \).
- The sequence \( \{r^n\}_{n=1}^{\infty} \) diverges to \( \infty \) if and only if \( r \) satisfies \( \ldots \).
- The sequence \( \{r^n\}_{n=1}^{\infty} \) diverges but does not diverge to \( \infty \) if and only if \( r \) satisfies \( \ldots \).

**0b.** Fix \( r \in \mathbb{R} \) with \( r \neq 1 \). For \( N \geq 1700 \), let \( s_N = \sum_{n=1700}^{N} r^n \). Note the sum starts at 1700. Then \( s_N \) can be written as:

\[
s_N = \ldotsan

\]for all \( N \geq 1700 \). Your answer should NOT contain a “…” nor a “\( \sum \)” sign.

**0c.** State the \( n^{\text{th}} \)-term test for an arbitrary series \( \sum a_n \).

**0d. Geometric Series.** Fill in the boxes with the proper range of \( r \in \mathbb{R} \). (Hint: look at the previous questions.)

- The series \( \sum r^n \) converges if and only if \( r \) satisfies \( \ldots \).
- The series \( \sum r^n \) diverges if and only if \( r \) satisfies \( \ldots \).

**0e. \( p \)-series.** Fill in the boxes with the proper range of \( p \in \mathbb{R} \).

- The series \( \sum \frac{1}{n^p} \) converges if and only if \( \ldots \).
- The series \( \sum \frac{1}{n^p} \) diverges if and only if \( \ldots \).

**Tests for Positive-Termed Series**

(\( \text{so for } \sum a_n \text{ where } a_n \geq 0 \))

**0f.** State the **Integral Test** for a positive-termed series \( \sum a_n \).

Let \( f : [1, \infty) \to \mathbb{R} \) be so that

- \( a_n = f \left( \ldots \right) \) for each \( n \in \mathbb{N} \)
- \( f \) is a function
- \( f \) is a function
- \( f \) is a function

Then \( \sum a_n \) converges if and only if \( \ldots \) converges.

**0g.** State the **Comparison Test** for a positive-termed series \( \sum a_n \).

Let \( N_0 \in \mathbb{N} \).

- If \( \ldots \) when \( n \geq N_0 \) and \( \ldots \), then \( \sum a_n \) converges.
- If \( \ldots \) when \( n \geq N_0 \) and \( \ldots \), then \( \sum a_n \) diverges.

Hint: sing the song to yourself.

**0h.** State the **Limit Comparison Test** for a positive-termed series \( \sum a_n \).

Let \( b_n > 0 \) and \( L = \lim_{n \to \infty} \ldots \).

- If \( \ldots \), then \( \ldots \).
- If \( \ldots \), then \( \ldots \).
- If \( \ldots \), then \( \ldots \).

Goal: cleverly pick positive \( b_n \)’s so that you know what \( \sum b_n \) does (converges or diverges) and the sequence \( \{ \frac{a_n}{b_n} \}_{n=1}^{\infty} \) converges.
Claim 1: If $x > 0$, then  

$$\ln x \leq x^1 \leq e^x.$$ 

To see this, consider the function $g(x) = e^x - x$. Then $g(0) = 1$ and $g'(x) = e^x > 0$ for $x > 0$. So $g(x) > 0$ for $x > 0$. Recall that the graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ over the line $y = x$.

Claim 2: Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \geq N_q$ then 

$$\ln x \leq x^q \leq e^x.$$ 

To see Claim 2, use L'Hôpital's rule to show that 

$$\lim_{x \to \infty} \frac{\log_b x}{x^q} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{x^q}{e^x} = 0.$$ 

(∗) 

Recall that $\log_b x = \ln x$. Recall that for any base $b > 0$ with $b \neq 1$ 

$$\log_b x = \frac{\log_e x}{\log_e b} \quad \text{and} \quad D_x \log_b x = \frac{1}{x \ln b} \quad \text{and} \quad D_x b^x = b^x \ln b$$ 

and $\lim_{x \to \infty} b^x = \infty$ if and only if $b > 1$. And so (∗) holds if one replaces $e$ with any base $b > 1$.

Claim 3: Consider a positive power $q > 0$ along with a base $b > 1$. There is (some big #) $N_{q,b} > 0$ so that if $x \geq N_{q,b}$ then 

$$\log_b x \leq x^q \leq b^x.$$ 

Moral: To figure out what is happening to a series involving $\log_b n$ or $b^n$, keep in mind that as $n \to \infty$ 

- $\log_b n$ grows super slow compared to $n^q$ 
- $b^n$ grows super fast compared to $n^q$

for any positive power $q > 0$ and base $b > 1$. 