Example 5.	$r = 2\sin(\theta)$	Find the Arc Length!

Note, letting θ range from 0 to 2π traces the curve (here it is a circle) TWICE.

We get the curve traced ONCE by letting θ range from 0 to π.

Next calculate:

$$\frac{dr}{d\theta} := \frac{d}{d\theta}\big(2\sin(\theta)\big) = 2\cos(\theta)$$

Now for the Arc Length:

$$\text{Arc Length} = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sqrt{\big(2\sin(\theta)\big)^2 + \big(2\cos(\theta)\big)^2}\, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sqrt{4\left(\sin^2\theta + \cos^2\theta\right)}\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sqrt{4\,(1)}\, d\theta \quad = \quad \int_{\theta=0}^{\theta=\pi} 2\, d\theta$$

$$= 2\theta\Big|_{\theta=0}^{\theta=\pi} \; = \; 2\,(\pi - 0) = \boxed{2\pi}\,.$$

The arc length of a circle is just the circumference of the circle.

The circumference of a circle with radius r is $2\pi\, r$.

Our circle $r = 2\sin\theta$ has radius 1 and so it's circumference is $2\pi \cdot 1 = 2\pi$.

Example 6. Express the arc length of $\boxed{r = 2 + 2\sin\theta}$ as an integral with respect to θ.

We trace the curve ONCE by letting θ range from 0 to 2π.

$$\begin{aligned}
\text{Arc Length} &= \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \sqrt{(2+2\sin\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta
\end{aligned}$$

next calculate: $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}(2+2\sin\theta) = 0 + 2\cos\theta = 2\cos\theta$ and so we get

$$= \boxed{\int_{\theta=0}^{\theta=2\pi} \sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta}$$

and using symmetry, we can also express as

$$\begin{aligned}
&= 2\left[\text{arc length in the } 4^{\text{th}} \text{ and } 1^{\text{st}} \text{ Quadrants}\right] \\
&= 2\left[\int_{\theta=\frac{3\pi}{2}}^{\theta=2\pi} \sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta + \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta\right] \\
&= 2\left[\int_{\theta=-\frac{\pi}{2}}^{\theta=0} \sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta + \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta\right] \\
&= \boxed{2\left[\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta\right]}\,.
\end{aligned}$$

Example 7. Express the arc length of the little loop of $\boxed{r = 1 + 2\cos\theta}$ as an integral with respect to θ.

To trace just the "little loop" of the curve we let θ range from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$.

Next calculate:

$$\frac{dr}{d\theta} := \frac{d}{d\theta}\big(1 + 2\cos(\theta)\big) = 0 + (-1)2\sin(\theta) = -2\sin(\theta)$$

Now for the Arc Length:

$$\text{Arc Length} = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\mathrm{d}\theta$$

$$= \boxed{\int_{\theta=\frac{2\pi}{3}}^{\theta=\frac{4\pi}{3}} \sqrt{\Big(1 + 2\cos(\theta)\Big)^2 + \big(-2\sin(\theta)\big)^2}\,\mathrm{d}\theta}\,.$$