In this handout: $\sum a_n$ is an arbitrary-termed series (i.e. $-\infty < a_n < \infty$).

Definitions

$\sum a_n$ is absolutely convergent $\iff$ $\left[\sum |a_n| \text{ converges}\right]$

$\sum a_n$ is conditionally convergent $\iff$ $\left[\sum |a_n| \text{ diverges and } \sum a_n \text{ converges}\right]$

$\sum a_n$ is divergent $\iff$ $\left[\sum a_n \text{ diverges}\right]$

Summarizing:

By definition, $\sum a_n$ is		$\sum \lvert a_n\rvert$		$\sum a_n$
absolutely convergent	if and only if	converges		
conditionally convergent	if and only if	diverges	and	converges
divergent	if and only if			diverges

Big Important Theorem: Absolute Convegence ⇒ Convergence

If $\sum |a_n|$ converges , then $\sum a_n$ converges .

So we get for free:

If $\sum a_n$ diverges , then $\sum |a_n|$ diverges .

Combining the Definition and Big Important Theorem we get

If $\sum a_n$ is		$\sum \lvert a_n\rvert$		$\sum a_n$
absolutely convergent	then	converges	$\overset{\text{so get}}{\Longrightarrow}$	converges
conditionally convergent	then	diverges	and	converges
divergent	then	diverges	$\overset{\text{so get}}{\Longleftarrow}$	diverges

So each arbitrary-termed series $\sum a_n$ is one, and only one, of the three possibilities:

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

Ratio Test & Root Test (for an arbitrary-termed series $\sum a_n$).

For the Ratio Test, set $$\rho := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$

For the Root Test, set $$\rho := \lim_{n\to\infty} \sqrt[n]{|a_n|} \overset{\text{note}}{=} \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

Then

$0 \le \rho < 1 \implies \sum a_n$ converges absolutely

$\rho = 1 \implies$ test is inconclusive (the test doesn't tell us anything)

$1 < \rho \le \infty \implies \sum a_n$ diverges (by the n$^{\text{th}}$-term test for divergence) .

Alternating Series Test (AST) & AST Remainder Estimate

Let

(1) $u_n \geq 0$ for each $n \in \mathbb{N}$

(2) $\lim_{n\to\infty} u_n = \boxed{0}$

(3) $u_n \boxed{> \text{ (also ok } \geq)} u_{n+1}$ for each $n \in \mathbb{N}$.

Then

- the series $\sum(-1)^n u_n$ converges. (also ok: $\sum(-1)^{n+1}u_n$ converges or $\sum(-1)^{n-1}u_n$ converges)
- and we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty}(-1)^n u_n$ by the finite sum $\sum_{k=1}^{N}(-1)^k u_k$ and the error (i.e. remainder) satisfies

$$\left|\sum_{k=1}^{\infty}(-1)^k u_k - \sum_{k=1}^{N}(-1)^k u_k\right| \leq \boxed{u_{N+1}} .$$

►. How to Remember the AST Remainder Estimate.
Let $u_n \searrow 0$ (i.e. $\{u_n\}_{n=1}^{\infty}$ satisifies conditions (1) – (3) of the AST) and

$$\sum_{n=1}^{\infty} (1)^{n+1} u_n = L . \tag{1}$$

Consider the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of the series in (**??**):

$$s_n = \sum_{k=1}^{n} (1)^{k+1} u_k .$$

So

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 - u_2 \\ s_3 &= u_1 - u_2 + u_3 \\ s_4 &= u_1 - u_2 + u_3 - u_4 \\ &\vdots \end{aligned}$$

and

$$\lim_{n\to\infty} s_n = L .$$

Then we have the following scenario. (Thomas book page 612.)

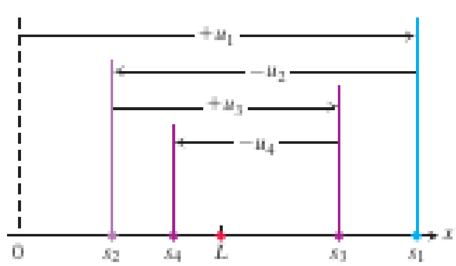

FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.