Def. An **alternating series** is of the form, where

$$u_n > 0$$

either

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 \pm \ldots \quad \stackrel{\text{i.e.}}{=} \quad \sum_{n=1}^{\infty} (-1)^{n+1}\, u_n \qquad (1)$$

or

$$-\, u_1 + u_2 - u_3 + u_4 - u_5 \pm \ldots \quad \stackrel{\text{i.e.}}{=} \quad \sum_{n=1}^{\infty} (-1)^{n}\, u_n \qquad (2)$$

BTW: Thomas's book defines an alternating series by (1) but should have said either (1) or (2).

Q.1. Question 1: Are these series alternating series? (Yes/No)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=17}^{\infty} \frac{(-1)^n}{n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{[\cos(\pi n)]^n}{n}$

Alternating Series Test (AST)

AST. Let

(1) $u_n > 0$ for each $n \in \mathbb{N}$
(2) $\lim_{n\to\infty} u_n = 0$
(3) $u_n > u_{n+1}$ for each $n \in \mathbb{N}$.

Then the alternating series $\sum (-1)^n u_n$ converges.

Note that then the alternating series $\sum (-1)^{n+1} u_n$ also converges (since $\sum (-1)^{n+1} u_n = (-1)^1 \sum (-1)^n u_n$).

BTW: Conditions (1) – (3) are often summarized by $u_n \searrow 0$.

BTW: When testing if an alternating series $\sum (-1)^n u_n$ is abs. conv., think of $\sum |(-1)^n u_n| \stackrel{\text{as}}{=} \sum a_n$ with $|a_n| = u_n$.

Q.2. Question 2: What if trying the AST on $\sum (-1)^n u_n$ and get

(1) $u_n > 0$ for each $n \in \mathbb{N}$
(2) $\lim_{n\to\infty} u_n = 17$.

So we cannot apply the AST ☹☹.

But all is not lost ☺☺.

What test can we now call upon?

Example 1.

Below the choice-boxes (AC/CC/Divg), carefully justify the given series's behavior. Be sure to specify which test(s) you are using and clearly explain your logic. Then check the correct choice-box.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

1.1. Is $\sum \frac{(-1)^n}{n}$ an alternating series?

1.2. Does $\sum \left| \frac{(-1)^n}{n} \right|$ converge?

1.3. Does $\sum \frac{(-1)^n}{n}$ converge?

1.4. Conculsion.

Example 2.

Below the choice-boxes (AC/CC/Divg), carefully justify the given series's behavior. Be sure to specify which test(s) you are using and clearly explain your logic. Then check the correct choice-box.

$$\sum_{n=1}^{\infty} (-1)^n \, \frac{n^3}{n^4+1}$$

☐ absolutely convergent

☐ conditionally convergent

☐ divergent

2.1. Thinking Land

2.2. Does $\sum \left| (-1)^n \, \frac{n^3}{n^4+1} \right|$ converge?

2.3. Does $\sum (-1)^n \, \frac{n^3}{n^4+1}$ converge?

2.4. Conculsion.

How to think about (i.e. how to remember) AST and Remainder

Let $\{u_n\}_{n=1}^{\infty}$ satisfy the conditions of the Alternating Series Test, i.e.,

(1) $u_n > 0$ and (2) $\lim_{n\to\infty} u_n = 0$ and (3) $u_n > u_{n+1}$.

(BTW: Conditions (1) – (3) are often summarized by $u_n \searrow 0$)

Consider the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of the series

$$s_n = \sum_{k=1}^{n} (1)^{k+1} u_k \ .$$

So

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 - u_2 \\ s_3 &= u_1 - u_2 + u_3 \\ s_4 &= u_1 - u_2 + u_3 - u_4 \\ &\vdots \end{aligned}$$

Then we have the following scenario. (Thomas book page 612.)

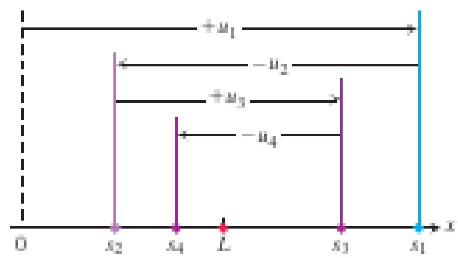

FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

Note that for $s_n = \sum_{k=1}^{n} (1)^{k+1} u_k$,

$$s_2 \le s_4 \le s_6 \le \ldots \le \ldots \le s_5 \le s_3 \le s_1$$

and there $L \in \mathbb{R}$ so that

$$\lim_{n\to\infty} s_n = L \ , \qquad \text{which just means that} \qquad \sum_{n=1}^{\infty} (1)^{k+1} u_k = L \ .$$

Key idea behind Reminder: What is the biggest the distance between s_2 and L be?
Well, ... $\text{dist}\,(s_2, L) < \text{dist}\,(s_2, s_3) = u_3$.

AST and Remainder

Let $\{u_n\}_{n=1}^{\infty}$ satisfy the **3** conditions of the Alternating Series Test, i.e., $u_n \searrow 0$).
Then alternating series $\sum_{k=1}^{\infty}(-1)^k u_k$ coverges and we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty}(-1)^n u_n$ by the finite sum $\sum_{k=1}^{N}(-1)^k u_k$ and the error (i.e. remainder) satisfies
(e.g., think about

$$\left| \sum_{k=1}^{\infty}(-1)^k u_k - \sum_{k=1}^{N}(-1)^k u_k \right| < \boxed{u_{N+1}} \ .$$

We will learn later in this Chapter that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{,for any } x \in \mathbb{R},$$

so taking $x = -1$ gives us that

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{4!} \pm \ldots$$

So (note where we start the sum ... cancellation going on here ...)

$$\boxed{e^{-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}}, \tag{1.1}$$

which we will use in the next two examples.

Example 3.

Estimte the error made by using the first 4 terms of the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$ to approximate e^{-1}.

Recall in (1.1) we say that

$$\boxed{e^{-1} \;=\; \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}}, \tag{1.1}$$

Example 4.
How large should we take N as to be guaranteed that

$$\left| e^{-1} \;-\; \sum_{n=2}^{N} \frac{(-1)^n}{n!} \right| \;\leq\; 0.01? \tag{4.1}$$

Note using (1.1) and the fact $0.01 \;=\; \frac{0.01}{1.00} \;=\; \frac{001}{100} \;=\; \frac{1}{100}$, the inequality (4.1) can be written as

$$\left| \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \;-\; \sum_{n=2}^{N} \frac{(-1)^n}{n!} \right| \;\leq\; \frac{1}{100}? \tag{4.1$'$}$$

Justify your answer with an appropriate infinite series remainder test.
Let's recall . . . so far we have remainder theorems for the Integral Test and AST.