In this handout: $\sum a_n$ is an arbitrary-termed series (i.e. $-\infty < a_n < \infty$).

## Definitions

$$\sum a_n \text{ is absolutely convergent} \iff \left[\sum |a_n| \text{ converges}\right]$$

$$\sum a_n \text{ is conditionally convergent} \iff \left[\sum |a_n| \text{ diverges and } \sum a_n \text{ converges}\right]$$

$$\sum a_n \text{ is divergent} \iff \left[\sum a_n \text{ diverges}\right]$$

Summarizing:

| By definition, $\sum a_n$ is | | $\sum \lvert a_n\rvert$ | | $\sum a_n$ |
|---|---|---|---|---|
| absolutely convergent | if and only if | converges | | |
| conditionally convergent | if and only if | diverges | and | converges |
| divergent | if and only if | | | diverges |

## Big Important Theorem

$$\text{If } \sum |a_n| \text{ converges, then } \sum a_n \text{ converges.}$$

So we get for free:

$$\text{If } \sum a_n \text{ diverges, then } \sum |a_n| \text{ diverges.}$$

| Combining the Definition and Big Important Theorem we get | | | | |
|---|---|---|---|---|
| If $\sum a_n$ is | | $\sum \lvert a_n\rvert$ | | $\sum a_n$ |
| absolutely convergent | then | converges | $\overset{\text{so get}}{\Longrightarrow}$ | converges |
| conditionally convergent | then | diverges | and | converges |
| divergent | then | diverges | $\overset{\text{so get}}{\Longleftarrow}$ | diverges |

So each arbitrary-termed series $\sum a_n$ is one, and only one, of the three possibilities:

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

## Ratio Test & Root Test (for an arbitrary-termed series $\sum a_n$).

For the Ratio Test, set $$\rho := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$

For the Root Test, set $$\rho := \lim_{n\to\infty} \sqrt[n]{|a_n|} \overset{\text{note}}{=} \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

Then

$0 \le \rho < 1 \implies \sum a_n$ converges absolutely

$\rho = 1 \implies$ test is inconclusive (the test doesn't tell us anything)

$1 < \rho \le \infty \implies \sum a_n$ diverges (by the $n^{\text{th}}$-term test for divergence) .