In this handout: $\sum a_n$ is an arbitrary-termed series (i.e. $-\infty < a_n < \infty$).

Definitions

$$\sum a_n \text{ is } \underline{\text{absolutely convergent}} \iff \left[\sum |a_n| \text{ converges}\right]$$

$$\sum a_n \text{ is } \underline{\text{conditionally convergent}} \iff \left[\sum |a_n| \text{ diverges and } \sum a_n \text{ converges}\right]$$

$$\sum a_n \text{ is } \underline{\text{divergent}} \iff \left[\sum a_n \text{ diverges}\right]$$

Summarizing:

By definition, $\sum a_n$ is		$\sum \lvert a_n \rvert$		$\sum a_n$
absolutely convergent	if and only if			
conditionally convergent	if and only if		and	
divergent	if and only if			

Big Important Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

So we get for free:

If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.

Combining the Definition and Big Important Theorem we get				
If $\sum a_n$ is		$\sum \lvert a_n \rvert$		$\sum a_n$
absolutely convergent	then		$\xRightarrow{\text{so get}}$	
conditionally convergent	then		and	
divergent	then		$\xLeftarrow{\text{so get}}$	

So each arbitrary-termed series $\sum a_n$ is one, and only one, of the three possibilities:

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

Ratio Test & Root Test (for an arbitrary-termed series $\sum a_n$).

For the Ratio Test, set $\rho := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

For the Root Test, set $\rho := \lim_{n\to\infty} \sqrt[n]{|a_n|} \overset{\text{note}}{=} \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.

Then

$0 \leq \rho < 1 \implies \sum a_n$ converges absolutely

$\rho = 1 \implies$ test is inconclusive (the test doesn't tell us anything)

$1 < \rho \leq \infty \implies \sum a_n$ diverges (by the n^{th}-term test for divergence).

Consider $\underbrace{0 < u_n \downarrow 0}_{\text{Conditions of AST}}$ so (by AST) $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.

Let $\sum_{n=1}^{\infty} (-1)^{n} u_n = L$.

Thomas, 13ed ET

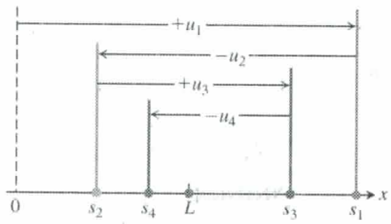

FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

$$\left|\sum_{n=1}^{\infty} (-1)^{n+1} u_n - \sum_{n=1}^{3} (-1)^{n+1} u_n\right|$$

$$= |L - s_3|$$

= length of

go up & draw

note

$\leq u_4$

integer n. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is nonincreasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $u_n \geq u_{n+1}$, for $n \geq N$.

EXAMPLE 2 Consider the sequence where $u_n = 10n/(n^2 + 16)$. Define $f(x) = 10x/(x^2 + 16)$. Then from the Derivative Quotient Rule,

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0 \qquad \text{whenever } x \geq 4.$$

It follows that $u_n \geq u_{n+1}$ for $n \geq 4$. That is, the sequence $\{u_n\}$ is nonincreasing for $n \geq 4$. ■

A graphical interpretation of the partial sums (Figure 10.13) shows how an alternating series converges to its limit L when the three conditions of Theorem 15 are satisfied with $N = 1$. Starting from the origin of the x-axis, we lay off the positive distance $s_1 = u_1$. To find the point corresponding to $s_2 = u_1 - u_2$, we back up a distance equal to u_2. Since $u_2 \leq u_1$, we do not back up any farther than the origin. We continue in this seesaw fashion, backing up or going forward as the signs in the series demand. But for $n \geq N$, each forward or backward step is shorter than (or at most the same size as) the preceding step because $u_{n+1} \leq u_n$. And since the nth term approaches zero as n increases, the size of step we take forward or backward gets smaller and smaller. We oscillate across the limit L, and the amplitude of oscillation approaches zero. The limit L lies between any two successive sums s_n and s_{n+1} and hence differs from s_n by an amount less than u_{n+1}.

Because

$$|L - s_n| < u_{n+1} \qquad \text{for } n \geq N,$$

we can make useful estimates of the sums of convergent alternating series.

THEOREM 16—The Alternating Series Estimation Theorem If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1} u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

We leave the verification of the sign of the remainder for Exercise 61.

EXAMPLE 3 We try Theorem 16 on a series whose sum we know:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} \,\Big|\, + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is $s_8 = 0.6640625$ and the sum of the first nine terms is $s_9 = 0.66796875$. The sum of the geometric series is

$$\frac{1}{1 - (-1/2)} = \frac{1}{3/2} = \frac{2}{3},$$

and we note that $0.6640625 < (2/3) < 0.66796875$. The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and is less than $(1/256) = 0.00390625$. ■