Fix an interval I in the real line (e.g., I might be $(-17, 19)$) and let x_0 be a point in I, i.e.,

$$x_0 \in I \ .$$

Next consider a function, whose domain is I,

$$f \colon I \to \mathbb{R}$$

and whose derivatives $f^{(n)} \colon I \to \mathbb{R}$ exist on the interval I for $n = 1, 2, 3, \ldots, N$.

Definition 1. The *N^{th}-order Taylor polynomial* for $y = f(x)$ at x_0 is:

$$p_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N \ , \qquad \text{(open form)}$$

which can also be written as (recall that $0! = 1$)

$$p_N(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N \quad \hookleftarrow \text{a finite sum, i.e. the sum stops} \ .$$

Formula (open form) is in open form. It can also be written in closed form, by using sigma notation, as

$$p_N(x) \ = \ \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} \ (x - x_0)^n \ . \qquad \text{(closed form)}$$

So $y = p_N(x)$ is a polynomial of degree at most N and it has the form

$$p_N(x) \ = \ \sum_{n=0}^{N} c_n \ (x - x_0)^n \qquad \text{where the constants} \qquad c_n \ = \ \frac{f^{(n)}(x_0)}{n!}$$

are specially chosen so that derivatives match up at x_0, i.e. the constants c_n's are chosen so that:

$$\begin{aligned} p_N(x_0) &= f(x_0) \\ p_N^{(1)}(x_0) &= f^{(1)}(x_0) \\ p_N^{(2)}(x_0) &= f^{(2)}(x_0) \\ &\vdots \\ p_N^{(N)}(x_0) &= f^{(N)}(x_0) \ . \end{aligned}$$

The constant c_n is the n^{th} Taylor coefficient of $y = f(x)$ about x_0. The *N^{th}-order Maclaurin polynomial* for $y = f(x)$ is just the N^{th}-order Taylor polynomial for $y = f(x)$ at $x_0 = 0$ and so it is

$$p_N(x) \ = \ \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \ x^n \ .$$

Definition 2. [1] The *Taylor series* for $y = f(x)$ at x_0 is the power series:

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad \text{(open form)}$$

which can also be written as

$$P_\infty(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots \quad \hookleftarrow \text{the sum keeps on going and going.}$$

The Taylor series can also be written in closed form, by using sigma notation, as

$$P_\infty(x) \ = \ \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \ (x - x_0)^n \ . \qquad \text{(closed form)}$$

The *Maclaurin series* for $y = f(x)$ is just the Taylor series for $y = f(x)$ at $x_0 = 0$.

[1] Here we are assuming that the derivatives $y = f^{(n)}(x)$ exist for each x in the interval I and for each $n \in \mathbb{N} \equiv \{1, 2, 3, 4, 5, \ldots\}$.

Easier Question 3. For what values of x does the power (a.k.a. Taylor) series

$$P_\infty(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \tag{1}$$

converge? The Root or Ratio test usually answers this question for us.

Big Question 4. If the power/Taylor series in formula (1) does indeed converge at a point x, i.e., if

$$\lim_{N\to\infty} P_N(x) \qquad \text{exists} \;,$$

does the series converge to what we would want it to converge to, i.e., does

$$\lim_{N\to\infty} P_N(x) \;\stackrel{?}{=}\; f(x) \tag{2}$$

in short, we are asking

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \;\;?$$

The question in (2) is going to take some thought.

Definition 5. The <u>N^{th}*-order Remainder term*</u> for $y = f(x)$ at x_0 is:

$$R_N(x) \;\stackrel{\text{def}}{=}\; f(x) - P_N(x)$$

where $y = P_N(x)$ is the N^{th}-order Taylor polynomial for $y = f(x)$ at x_0.

So

$$f(x) \;=\; P_N(x) + R_N(x) \tag{3}$$

that is

$$f(x) \;\approx\; P_N(x) \qquad \text{within an error of} \quad R_N(x)\;.$$

The question is

$$f(x) \;\stackrel{??}{=}\; P_\infty(x) \qquad \text{i.e.,} \qquad f(x) \;\stackrel{??}{=}\; \lim_{N\to\infty} P_N(x)\;,$$

where $y = P_\infty(x)$ is the Taylor series of $y = f(x)$ at x_0.
Well, let's think about what needs to be for $f(x) \stackrel{??}{=} P_\infty(x)$, i.e., <u>for f to equal to its Taylor series</u>.

Notice 6. Taking the $\lim_{N\to\infty}$ of both sides in equation (3), we see that

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \qquad \hookleftarrow \text{the sum keeps on going and going}\;.$$

if and only if

$$\lim_{N\to\infty} R_N(x) \;=\; 0\;.$$

Recall 7. $\lim_{N\to\infty} R_N(x) \;=\; 0$ if and only if $\lim_{N\to\infty} |R_N(x)| \;=\; 0$.

<u>**Answer to the Big Question 4.**</u> So we know see that the following are equivalent,
(i.e. (4) holds $\Longleftrightarrow$ (5) holds $\Longleftrightarrow$ (6) holds).

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n\;. \tag{4}$$

$$\lim_{N\to\infty} R_N(x) \;=\; 0\;. \tag{5}$$

$$\lim_{N\to\infty} |R_N(x)| \;=\; 0\;. \tag{6}$$

So, to show that a function is equal to its Taylor series, we basically want to show that (6) holds true.
How to do this? Well, this is where Mr. Taylor comes to the rescue![2]

[2]According to Mr. Taylor, his Remainder Theorem (see next page) was motivated by coffeehouse conversations about works of Newton on planetary motion and works of Halley (of *Halley's comet*) on roots of polynomials.

Taylor's Remainder Theorem

Taylor's Remainder Theorem. Fix a point $x \in I$ and fix $N \in \mathbb{N}$. [3]

There exists c between x and x_0 so that

$$R_N(x) \overset{\text{def}}{=} f(x) - P_N(x) \overset{\text{theorem}}{=} \frac{f^{(N+1)}(c)}{(N+1)!} (x - x_0)^{(N+1)} . \tag{7}$$

So either $x \le c \le x_0$ or $x_0 \le c \le x$. So we do not know exactly what c is but at least we know that c is between x and x_0 and so $c \in I$.

Remark: This is a Big Theorem by Taylor. See the book for the proof. The proof uses the Mean Value Theorem.

Note that formula (7) implies that

$$|R_N(x)| = \frac{\left|f^{(N+1)}(c)\right|}{(N+1)!} |x - x_0|^{(N+1)} . \tag{8}$$

How we often use Taylor's Remainder Theorem. Fix an interval I and fix $N \in \mathbb{N}$.[3]

Assume we can find $M \in \mathbb{R}$ so that

$$\text{the maximum of } \left|f^{(N+1)}(x)\right| \text{ on the interval } I \le M ,$$

i.e.,

$$\max_{c \in I} \left|f^{(N+1)}(c)\right| \le M .$$

Then

$$|R_N(x)| \le \frac{M}{(N+1)!} |x - x_0|^{N+1} \tag{9}$$

for each $x \in I$.

Remark: This follows from formula (8).

[3] Here we assume that the $(N+1)$-derivative of $y = f(x)$, i.e. $y = f^{(N+1)}(x)$, exists for each $x \in I$.