Commonly Used Taylor Series

<table>
<thead>
<tr>
<th>SERIES</th>
<th>WHEN IS VALID/TRUE</th>
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</thead>
<tbody>
<tr>
<td>[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots ]</td>
<td>NOTE THIS IS THE GEOMETRIC SERIES. JUST THINK OF ( x ) AS ( r ) ( x \in (-1, 1) )</td>
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<tr>
<td>[ \sum_{n=0}^{\infty} x^n ]</td>
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<tr>
<td>[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots ]</td>
<td>SO: ( e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots ) ( e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \sum_{n=0}^{\infty} \frac{17^n x^n}{n!} )</td>
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<td>[ \sum_{n=0}^{\infty} \frac{x^n}{n!} ]</td>
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<td>[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots ]</td>
<td>NOTE ( y = \cos x ) IS AN EVEN FUNCTION (i.e., ( \cos(-x) = \cos(x) )) AND THE TAYLOR SERIES OF ( y = \cos x ) HAS ONLY EVEN POWERS.</td>
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<tr>
<td>[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} ]</td>
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<td>[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots ]</td>
<td>NOTE ( y = \sin x ) IS AN ODD FUNCTION (i.e., ( \sin(-x) = -\sin(x) )) AND THE TAYLOR SERIES OF ( y = \sin x ) HAS ONLY ODD POWERS.</td>
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<tr>
<td>[ \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} ]</td>
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<tr>
<td>[ \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots ]</td>
<td>QUESTION: IS ( y = \ln(1 + x) ) EVEN, ODD, OR NEITHER?</td>
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<td>[ \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} ]</td>
<td>( x \in (-1, 1) )</td>
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<tr>
<td>[ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots ]</td>
<td>QUESTION: IS ( y = \arctan(x) ) EVEN, ODD, OR NEITHER?</td>
</tr>
<tr>
<td>[ \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} ]</td>
<td>( x \in [-1, 1] )</td>
</tr>
</tbody>
</table>
Fix an interval $I$ in the real line (e.g., $I$ might be $(-17, 19)$) and let $x_0$ be a point in $I$, i.e.,

$$x_0 \in I.$$ 

Next consider a function, whose domain is $I$, $f: I \to \mathbb{R}$ and whose derivatives $f^{(n)}: I \to \mathbb{R}$ exist on the interval $I$ for $n = 1, 2, 3, \ldots, N$.

**Definition 1.** The $N^{th}$-order **Taylor polynomial** for $y = f(x)$ at $x_0$ is:

$$p_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N,$$

which can also be written as (recall that $0! = 1$)

$$p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

So $y = p_N(x)$ is a polynomial of degree at most $N$ and it has the form

$$p_N(x) = \sum_{n=0}^{N} c_n (x - x_0)^n$$

where the constants $c_n = \frac{f^{(n)}(x_0)}{n!}$ are specially chosen so that derivatives match up at $x_0$, i.e. the constants $c_n$’s are chosen so that:

$$p_N(x_0) = f(x_0)$$

$$p_N^{(1)}(x_0) = f^{(1)}(x_0)$$

$$p_N^{(2)}(x_0) = f^{(2)}(x_0)$$

$$\vdots$$

$$p_N^{(N)}(x_0) = f^{(N)}(x_0).$$

The constant $c_n$ is the $n^{th}$ Taylor coefficient of $y = f(x)$ about $x_0$. The $N^{th}$-order **Maclaurin polynomial** for $y = f(x)$ is just the $N^{th}$-order Taylor polynomial for $y = f(x)$ at $x = 0$ and so it is

$$p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n.$$

**Definition 2.** 1 The **Taylor series** for $y = f(x)$ at $x_0$ is the power series:

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$$

which can also be written as

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

The Taylor series can also be written in closed form, by using sigma notation, as

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

The **Maclaurin series** for $y = f(x)$ is just the Taylor series for $y = f(x)$ at $x_0 = 0$.

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1Here we are assuming that the derivatives $y = f^{(n)}(x)$ exist for each $x$ in the interval $I$ and for each $n \in \mathbb{N} \equiv \{1, 2, 3, 4, 5, \ldots\}$. 

2
Easier Question 3. For what values of $x$ does the power (a.k.a. Taylor) series
\[ P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \] (1)
converge? The Root or Ratio test usually answers this question for us.

Big Question 4. If the power/Taylor series in formula (1) does indeed converge at a point $x$, i.e., if
\[ \lim_{N \to \infty} P_N(x) \text{ exists}, \]
does the series converge to what we would want it to converge to, i.e., does
\[ \lim_{N \to \infty} P_N(x) = f(x) \] (2)
in short, we are asking
\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n ? \]
The question in (2) is going to take some thought.

Definition 5. The $N^{th}$-order Remainder term for $y = f(x)$ at $x_0$ is:
\[ R_N(x) \overset{\text{def}}{=} f(x) - P_N(x) \]
where $y = P_N(x)$ is the $N^{th}$-order Taylor polynomial for $y = f(x)$ at $x_0$.
So
\[ f(x) = P_N(x) + R_N(x) \] (3)
that is
\[ f(x) \approx P_N(x) \text{ within an error of } R_N(x). \]
The question is
\[ f(x) \overset{?}{=} P_\infty(x) \text{ i.e., } f(x) \overset{?}{=} \lim_{N \to \infty} P_N(x), \]
where $y = P_\infty(x)$ is the Taylor series of $y = f(x)$ at $x_0$.
Well, let’s think about what needs to be for $f(x) \overset{?}{=} P_\infty(x)$, i.e., for $f$ to equal to its Taylor series.

Notice 6. Taking the $\lim_{N \to \infty}$ of both sides in equation (3), we see that
\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \leftarrow \text{the sum keeps on going and going}. \]
if and only if
\[ \lim_{N \to \infty} R_N(x) = 0. \]

Recall 7. $\lim_{N \to \infty} R_N(x) = 0$ if and only if $\lim_{N \to \infty} |R_N(x)| = 0$.

Answer to the Big Question 4. So we know see that the following are equivalent,
(i.e. (4) holds $\iff$ (5) holds $\iff$ (6) holds).
\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \] (4)
\[ \lim_{N \to \infty} R_N(x) = 0. \] (5)
\[ \lim_{N \to \infty} |R_N(x)| = 0. \] (6)
So, to show that a function is equal to its Taylor series, we basically want to show that (6) holds true.
How to do this? Well, this is where Mr. Taylor comes to the rescue!\(^2\)

\(^2\)According to Mr. Taylor, his Remainder Theorem (see next page) was motivated by coffeehouse conversations about works of Newton on planetary motion and works of Halley (of Halley’s comet) on roots of polynomials.
**Taylor’s Remainder Theorem**

**Version 1**: for a fixed point $x \in I$ and a fixed $N \in \mathbb{N}$.

There exists $c$ between $x$ and $x_0$ so that

$$R_N(x) \overset{\text{def}}{=} f(x) - P_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x-x_0)^{(N+1)}.$$

So either $x \leq c \leq x_0$ or $x_0 \leq c \leq x$. So we do not know exactly what $c$ is but at least we know that $c$ is between $x$ and $x_0$ and so $c \in I$.

Remark: This is a **Big Theorem** by Taylor. See the book for the proof. The proof uses the Mean Value Theorem.

Note that formula (7) implies that

$$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} |x-x_0|^{(N+1)} \right|.$$ (8)

**Version 2**: for the whole interval $I$ and a fixed $N \in \mathbb{N}$.

Assume we can find $M$ so that the maximum of $\left| f^{(N+1)}(x) \right|$ on the interval $I$ $\leq M$.

i.e.,

$$\max_{c \in I} \left| f^{(N+1)}(c) \right| \leq M.$$

Then

$$|R_N(x)| \leq \frac{M}{(N+1)!} |x-x_0|^{N+1}$$ (9)

for each $x \in I$.

Remark: This follows from formula (8).

**Version 3**: for the whole interval $I$ and all $N \in \mathbb{N}$.

Now assume that we can find a sequence $\{M_N\}_{N=1}^{\infty}$ so that

$$\max_{c \in I} \left| f^{(N+1)}(c) \right| \leq M_N$$

for each $N \in \mathbb{N}$ and also so that

$$\lim_{N \to \infty} \frac{M_N}{(N+1)!} |x-x_0|^{N+1} = 0$$

for each $x \in I$. Then, by formula (9) and the Squeeze Theorem,

$$\lim_{N \to \infty} |R_N(x)| = 0$$

for each $x \in I$. Thus, by So,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

for each $x \in I$.

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3Here we assume that the $(N+1)$-derivative of $y = f(x)$, i.e. $y = f^{(N+1)}(x)$, exists for each $x \in I$.

4Here we assume that $y = f^{(N)}(x)$, exists for each $x \in I$ and each $N \in \mathbb{N}$. 

4