The integral test applies to positive-term series, i.e. infinite series $\sum a_n$ where each $a_n$ is positive. A main tool used in deciding if a positive-term series converges is the monotonic sequence theorem. Let’s recall the monotonic sequence theorem, which we learned while studying sequences.

**MONOTONIC SEQUENCE THEOREM**

If $\{x_n\}_{n \in \mathbb{N}}$ is an increasing sequence (i.e., $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$), then either
- $\{x_n\}_{n \in \mathbb{N}}$ is bounded above, in which case $\lim_{n \to \infty} x_n$ exists (as a finite real number)
- or
- $\{x_n\}_{n \in \mathbb{N}}$ is not bounded above, in which case $\lim_{n \to \infty} x_n = \infty$.

Recall: $\{x_n\}_{n \in \mathbb{N}}$ is bounded above means that there is a $B \in \mathbb{R}$ such that $x_n \leq B$ for each $n \in \mathbb{N}$.

**POSITIVE-TERM SERIES**

Definition: $\sum a_n$ is a positive-term series if $a_n \geq 0$ for each $n$.

Explore: Let $\sum a_n$ be a positive-term series.

1. Consider its sequence of partial sums $\{S_N\}_{N \in \mathbb{N}}$ where $S_N = a_1 + a_2 + \ldots + a_N$.
2. Recall that $\sum_{n=1}^{\infty} a_n \overset{\text{def}}{=} \lim_{N \to \infty} S_N$.
3. $[a_n \geq 0$ for each $n \in \mathbb{N}] \implies [S_N \leq S_{N+1}$ for each $N \in \mathbb{N}]$.
4. So $\{S_N\}_{N \in \mathbb{N}}$ is an increasing sequence.

So either:
- $\{S_N\}_{N \in \mathbb{N}}$ is bounded above, in which case, by the monotonic sequence theorem, $\lim_{N \to \infty} S_N$ exists (as a finite real number) and thus by (2) above, $\sum a_n$ converges (to a finite real number)
- or
- $\{S_N\}_{N \in \mathbb{N}}$ is not bounded above, in which case, by the monotonic sequence theorem, $\lim_{N \to \infty} S_N = \infty$ and thus by (2) above, $\sum a_n$ diverges (to $\infty$).

**TODAY’S GOAL**

Examine the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. We will show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent} & \text{if } p \leq 1 \end{cases}.$$

When $p = 1$, the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n}$ is also called the harmonic series.
**Integral Test**

Let’s say we are given a series $\sum a_n$ and we can find a function $f : [1, \infty) \to \mathbb{R}$ satisfying

1. $a_n = f(n)$ for each $n \in \mathbb{N}$ with $n \geq 1$ (this is usually accomplished by design)
2. $y = f(x)$ is positive on $[1, \infty)$ (so $\sum a_n$ needs to be a positive term series)
3. $y = f(x)$ is continuous on $[1, \infty)$
4. $y = f(x)$ is decreasing on $[1, \infty)$ (can confirm this by showing $f'(x) \leq 0$).

Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_{x=1}^{x=\infty} f(x) \, dx$ either:

- (a) both converge (to finite numbers, although most likely different numbers)
- (b) both diverge (to $\infty$).

**WHY THE INTEGRAL TEST IS TRUE**

Let’s say we are given a series $\sum a_n$ and find a function $f : [1, \infty) \to \mathbb{R}$ satisfying (1)–(4). Then $\{\sum_{n=1}^{\infty} a_n\}_{N \in \mathbb{N}}$ and $\{\int_{x=1}^{x=N} f(x) \, dx\}_{N \in \mathbb{N}}$ are both increasing sequences and so each has the choice of either [converging to some finite number] or [diverging to $\infty$]. Next note that

\[
\int_{x=1}^{x=N} f(x) \, dx \leq \sum_{n=1}^{N} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \leq \int_{x=1}^{x=\infty} f(x) \, dx
\]

Now take the limit as $N \to \infty$ to see that

\[
\sum_{n=2}^{\infty} a_n \quad \text{(A)} \quad \int_{x=1}^{x=\infty} f(x) \, dx \quad \text{by (B)} \quad \leq \sum_{n=1}^{\infty} a_n \quad \text{(*)}
\]

The integral test now follows from (\text{*}).

**Observation 1.** The statement of the Integral Test remains true if we replace each 1 with 17, or any other integer. This is useful if, e.g., you can get (1)–(3) to hold but only have $y = f(x)$ decreasing on $[17, \infty)$.

**Observation 2.** Let’s say that we have shown that $\sum a_n$ converges by using the integral test with the function $y = f(x)$, which satisfies that above conditions (1) - (4). Then we can approximate the infinite sum $S := \sum_{n=1}^{\infty} a_n$ by the computable finite sum $S_N := \sum_{n=1}^{N} a_n$. Indeed, define $S$, $S_N$, and $R_N$ by

\[
S := \sum_{n=1}^{\infty} a_n \quad \text{and} \quad S_N := \sum_{n=1}^{N} a_n \quad \text{and} \quad R_N := S - S_N \quad \text{; thus,} \quad S = S_N + R_N
\]

Then $S \approx S_N$ within an error of $|R_N|$ and

\[
0 \leq \int_{x=N+1}^{x=\infty} f(x) \, dx \quad \text{by (2)} \quad \sum_{n=N+1}^{\infty} a_n \quad \text{note} \quad R_N \quad \text{by (A)} \quad \leq \int_{x=N}^{x=\infty} f(x) \, dx
\]