# Formula/Concepts You Need To Know

## Review of some needed Trig. Identities for Integration

- Your answers should be an angle in **RADIANS**.
  - arccos( $\frac{1}{2}$ ) = $\frac{\pi}{3}$
  - arccos( -$\frac{1}{2}$ ) = $\frac{2\pi}{3}$
  - arcsin($\frac{1}{2}$) = $\frac{\pi}{6}$
  - arcsin( -$\frac{1}{2}$ ) = $\frac{-\pi}{6}$
  - Can you do similar problems?
- Double-angle formulas. Your answer should involve trig functions of $\theta$, and not of $2\theta$.
  - $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
  - $\sin(2\theta) = 2\sin\theta\cos\theta$ .
- Half-angle formulas. Your answer should involve $\cos(2\theta)$.
  - $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
  - $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$
- Since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween:
  - tangent (i.e., tan) and secant (i.e., sec) is $1+\tan^2\theta = \sec^2\theta$ .
  - cotangent (i.e., cot) and cosecant (i.e., csc) is $1+\cot^2\theta = \csc^2\theta$ .

## Remember Your Calculus I Integration Basics?

In this part, $a$ is a constant and $a > 0$.

- If $u \neq 0$, then $\int \frac{du}{u} = \ln|u| + C$
- If $a \neq 1$, then $\int a^u\,du = \frac{a^u}{\ln a} + C$
- $\int \cos u\,du = \sin u + C$
- $\int \sec^2 u\,du = \tan u + C$
- $\int \sec u \tan u\,du = \sec u + C$
- $\int \sin u\,du = -\cos u + C$
- $\int \csc^2 u\,du = -\cot u + C$
- $\int \csc u \cot u\,du = -\csc u + C$
- $\int \tan u\,du = \ln|\sec u| \overset{or}{=} -\ln|\cos u| + C$
- $\int \cot u\,du = -\ln|\csc u| \overset{or}{=} \ln|\sin u| + C$
- $\int \sec u\,du = \ln|\sec u + \tan u| \overset{or}{=} -\ln|\sec u - \tan u| + C$
- $\int \csc u\,du = -\ln|\csc u + \cot u| \overset{or}{=} \ln|\csc u - \cot u| + C$
- $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C$
- $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
- $\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

## Integration from Calculus II

- Partial Fraction Decomposition. Let's integrate $y = \frac{f(x)}{g(x)}$, where $f$ and $g$ are polyonomials, by 1st finding its PDF.
  - If [degree of $f$] $\geq$ [degree of $g$], then one must first do ___long division___.
  - If [degree of $f$] $<$ [degree of $g$], then first factor $y = g(x)$ into [linear] factors $px + q$ and irreducible [quadratic] factors $ax^2 + bx + c$ (to be sure it's irreducible, you need [$b^2 - 4ac < 0$]).
    Next, collect up like terms and follow the following rules.

**Rule 1**: For each factor of the form $(px + q)^m$ where $m \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of [$m$] partial fractions of the form, where each $A_i$ is a real number,

$$\boxed{\frac{A_1}{(px+q)^1} + \frac{A_2}{(px+q)^2} + \ldots + \frac{A_m}{(px+q)^m}}.$$

**Rule 2**: For each factor of the form $(ax^2 + bx + c)^n$ where $n \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of [$n$] partial fractions of the form, where the $A_i$'s and $B_i$'s are real number,

$$\boxed{\frac{A_1x+B_1}{(ax^2+bx+c)^1} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}}.$$

- Integration by parts formula: $\int u\,dv =$ ___$uv - \int vdu$___
- Trig. Substitution. (Recall that the *integrand* is the function you are integrating.) Here, $a$ is a constant and $a > 0$.
  - if the integrand involves $a^2 - u^2$, then one makes the substitution $u =$ ___$a\ \sin\theta$___.
  - if the integrand involves $a^2 + u^2$, then one makes the substitution $u =$ ___$a\tan\theta$___.
  - if the integrand involves $u^2 - a^2$, then one makes the substitution $u =$ ___$a\sec\theta$___.

## Sequences

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Complete the below sentences.
  - The limit of $\{a_n\}_{n=1}^{\infty}$ is the real number $L$ provided for each $\epsilon > 0$ there exists a natural number $N$ so that if the natural number $n$ satisfies ___$n$___ $>$ ___$N$___ then ___$|L - a_n|$___ $<$ ___$\epsilon$___.
  - If the limit of $\{a_n\}_{n=1}^{\infty}$ is $L \in \mathbb{R}$, then we denote this by ___$\lim_{n\to\infty} a_n = L$___.
  - $\{a_n\}_{n=1}^{\infty}$ converges provided ___there exists a real number $L$ so that $\lim_{n\to\infty} a_n = L$___.
  - $\{a_n\}_{n=1}^{\infty}$ diverges provided $\{a_n\}_{n=1}^{\infty}$ ___does not converge___.
  - $\{a_n\}_{n=1}^{\infty}$ diverges provided $\lim_{n\to\infty} a_n$ ___does not exist (i.e., DNE)___.
- Practice taking basic limits. (Important, e.g., for Ratio and Root Tests.)
  - $\lim\limits_{n\to\infty} \dfrac{5n^{17} + 6n^2 + 1}{7n^{18} + 9n^3 + 5} =$ ___0___
  - $\lim\limits_{n\to\infty} \dfrac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5} =$ ___$\frac{36}{4}$ or 9___
  - $\lim\limits_{n\to\infty} \dfrac{-5n^{18} + 6n^2 + 1}{7n^{17} + 9n^3 + 5} =$ ___DNE or $-\infty$___
  - $\lim\limits_{n\to\infty} \sqrt{\dfrac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5}} =$ ___$\sqrt{\frac{36}{4}}$ or 3___
  - Can you do similar problems?
- Let $-\infty < r < \infty$. (Needed for Geometric Series. Warning, don't confuse sequences with series.)
  - If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n =$ [0].
  - If $r = 1$, then $\lim\limits_{n\to\infty} r^n =$ [1].
  - If $r > 1$, then $\lim\limits_{n\to\infty} r^n =$ [DNE (tends to $\infty$)].
  - If $r = -1$, then $\lim\limits_{n\to\infty} r^n =$ [DNE (oscillates between 1 and $-1$)].
  - If $r < -1$, then $\lim\limits_{n\to\infty} r^n =$ [DNE ($r^{2n} \to \infty$ while $r^{2n+1} \to -\infty$)].

## Series

►. In this section, all series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

- For a formal <u>series</u> $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding <u>sequence</u> $\{s_N\}_{N=1}^{\infty}$ of partial sums, so $s_N = S\sum_{n=1}^{N} a_n$. Then the formal series $\sum a_n$
  - converges if and only if the $\lim_{N\to\infty} s_N$ exists
  - converges to $L \in \mathbb{R}$ if and only if the $\lim_{N\to\infty} s_N$ exists and equals $L$
  - diverges if and only if the $\lim_{N\to\infty} s_N$ does not exist .

  Now assume, furthermore, that $a_n \geq 0$ for each $n$. Then the <u>sequence</u> $\{s_N\}_{N=1}^{\infty}$ of partial sums either
  - <u>is</u> bounded above (by some finite number), in which case the <u>series</u> $\sum a_n$ converges

  or
  - <u>is not</u> bounded above (by some finite number), in which case the <u>series</u> $\sum a_n$ diverges .

- The $n^{\text{th}}$**-term test** for an arbitrary series $\sum a_n$.
  If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ [diverges] .

- Fix $r \in \mathbb{R}$. For $N \geq 17$, let $s_N = \sum_{\mathbf{n=17}}^{N} r^n$ (Note the sum starts at 17). Then, for $N > 17$,
  - $s_N =$ $r^{17} + r^{18} + \ldots + r^N$ (your answer can have ...'s but not a $\sum$ sign)
  - $r\, s_N =$ $r^{18} + \ldots + r^N + r^{N+1}$ (your answer can have ...'s but not a $\sum$ sign)
  - $(1-r)\, s_N =$ $r^{17} - r^{N+1}$ (your answer should have neither ...'s nor a $\sum$ sign)
  - and if $r \neq 1$, then $s_N =$ $\frac{r^{17} - r^{N+1}}{1-r}$ (your answer should have neither ...'s nor a $\sum$ sign)

- **Geometric Series** where $-\infty < r < \infty$. The series $\sum r^n$ (hint: look at the previous questions):
  - converges if and only if $|r|$ [$< 1$]
  - diverges if and only if $|r|$ [$\geq 1$] .

- **Integral Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$. Let $f\colon [1,\infty) \to \mathbb{R}$ be so that $a_n = f($[$n$]$)$ for each $n \in \mathbb{N}$ and $y = f(x)$ is a [continuous], [positive], [decreasing (nonincsreasing is also ok)] function. Then we have the following.
  1. For each $N > 2$,

$$\sum_{n=\boxed{2}}^{\boxed{N}} a_n \;\leq\; \int_{x=1}^{x=N} f(x)\,dx \;\leq\; \sum_{n=\boxed{1}}^{\boxed{N-1}} a_n \,. \qquad (1)$$

     Fill in, as so to give the best estimate one can, each of the 4 boxes with: a number, $N$, $N-1$, or $N+1$.

     Hint. Approximate (below and above) the $\int_1^N f(x)\,dx$ by the area of $N-1$ Riemann rectangles, each of base length $\Delta x = 1$ .
  2. From the bounds in (1), we see that $\sum a_n$ converges if and only if [$\int_{x=1}^{x=\infty} f(x)\,dx$] converges.

(3) Now let $\sum a_n$ converge. We want to approximate the infinite sum $\sum_{n=1}^{\infty} a_n$ by the finite sum $\sum_{n=1}^{N} a_n$ within an error (i.e., remainder) of $R_N$. To figure out how good this approximation is, define $R_N$ as below and get a good (as one can) lower and upper approximation of $R_N$, again using Reimann sums. Fill in the 3 boxes with: a number, $N$, $N-1$, or $N+1$ .

$$\boxed{0} \leq R_N \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=\boxed{N+1}}^{\infty} a_n \leq \int_{x=\boxed{N}}^{\infty} f(x)\, dx .$$

- **$p$-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$
  - converges if and only if $p$ $\boxed{> 1}$ .
  - diverges if and only if $p$ $\boxed{\leq 1}$ .

  This can be shown by using the <u>integral</u> test and comparing $\sum \frac{1}{n^p}$ to (the easy to compute) $\int_{x=1}^{\infty} \underline{\frac{1}{x^p}}\, dx$ .
- **Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$. (Fill in the blanks with $a_n$ and/or $b_n$.)
  - If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum \boxed{b_n}$ converge, then $\sum \boxed{a_n}$ converge.
  - If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum \boxed{b_n}$ diverge, then $\sum \boxed{a_n}$ diverge.

  Hint: sing the song to yourself.
- **Limit Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$. Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.
  If $\boxed{0} < L < \boxed{\infty}$, then $\sum a_n$ converges if and only if $\boxed{\sum b_n \text{ converges}}$ .
- **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

  - If $\rho$ $\boxed{< 1}$ then $\sum a_n$ converges absolutely.
  - If $\rho$ $\boxed{> 1}$ then $\sum a_n$ diverges.
  - If $\rho$ $\boxed{= 1}$ then the test is inconclusive (in other words, the test fails).
- **Alternating Series Test (AST) & Alternating Series Estimation Theorem (ASET).**
  Consider an alternating series $\sum (-1)^n u_n$ where $u_n > 0$ for each $n \in \mathbb{N}$.
  If
  - $u_n$ $\boxed{>}$ $u_{n+1}$ for each $n \in \mathbb{N}$
  - $\lim_{n\to\infty} u_n = \boxed{0}$

  then
  - $\sum (-1)^n u_n$ $\boxed{\text{converges}}$
  - we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty} (-1)^n u_n$ by the finite sum $\sum_{n=1}^{N} (-1)^n u_n$ and the error (i.e. remainder) satisfies

$$\left| \sum_{n=1}^{\infty} (-1)^n u_n - \sum_{n=1}^{N} (-1)^n u_n \right| \leq \boxed{u_{N+1}} .$$

- By definition, for an arbitrary series $\sum a_n$, (fill in these 4 boxes with converges or diverges).
  - $\sum a_n$ is <u>absolutely convergent</u> if and only if $\sum |a_n|$ $\boxed{\text{converges}}$ .
  - $\sum a_n$ is <u>conditionally convergent</u> if and only if $\sum a_n$ $\boxed{\text{converges}}$ and $\sum |a_n|$ $\boxed{\text{diverges}}$ .

- $\sum a_n$ is divergent if and only if $\sum a_n$ [ diverges ].

•. Fill in the 3 blank boxes with absolutely convergent, conditionally convergent, or divergent) on the following FLOW CHART from class used to determine the behavior of a series $\sum_{n=17}^{\infty} a_n$.

Does $\sum |a_n|$ converge? Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$ Does $\lim_{n\to\infty} |a_n| = 0$? if NO $\Rightarrow$ $\sum a_n$ is [ divergent ]

if YES $\Downarrow$ (from "Does $\sum |a_n|$ converge?"): $\sum a_n$ is [ absolutely convergent ]

if YES $\Downarrow$ (from "Does $\lim_{n\to\infty} |a_n| = 0$?"): Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is [ conditionally convergent ]

## Power Series

Condsider a (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \,(x - x_0)^n \;, \tag{2}$$

with radius of convergence $R \in [0, \infty]$. (Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

•. Fill in the next for boxes with one of the following 4 choices: [**a.**] is always absolutely convergent (AC) [**b.**] is always conditionally convergent (CC) [**c.**] is always divergent (DIV) [**d.**] can do anything, i.e., there are examples showing that it can be AC, CC, or DIV.

(1) For $x = x_0$, the power series $h(x)$ in (2) ___a___ .

(2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series $h(x)$ in (2) ___a___ .

(3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$ the power series $h(x)$ in (2) ___c___ .

(4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series $h(x)$ in (2) ___d___ .

•. For the next 2 problems, let $R > 0$ and fill-in the boxes. Consider the function $y = h(x)$ defined by the power series in (2).

(1) The function $y = h(x)$ is always differentiable on the interval [ $(x_0 - R, x_0 + R)$ ] (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

$$h'(x) \;=\; \sum_{n=1}^{\infty} \boxed{\; n\, a_n \,(x - x_0)^{n-1} \;} \,. \tag{3}$$

What can you say about the radius of convergence of the power series in (3)? [ It's the same $R$ ].

(2) The function $y = h(x)$ always has an antiderivative on the interval [ $(x_0 - R, x_0 + R)$ ] (make this interval as large as it can be, but still keeping the statement true). Futhermore, if $\alpha$ and $\beta$ are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\, dx \;=\; \sum_{n=0}^{\infty} \boxed{\; \frac{a_n}{n+1}\, (x - x_0)^{n+1} \;} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} \,.$$

## Taylor/Maclaurin Polynomials and Series

Let $y = f(x)$ be a function with derivatives of all orders in an interval $I$ containing $x_0$.

Let $y = P_N(x)$ be the $N^{\text{th}}$-order Taylor polynomial of $y = f(x)$ about $x_0$.

Let $y = R_N(x)$ be the $N^{\text{th}}$-order Taylor remainder of $y = f(x)$ about $x_0$.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about $x_0$.

Let $c_n$ be the $n^{\text{th}}$ Taylor coefficient of $y = f(x)$ about $x_0$.

**a.** The formula for $c_n$ is

$$c_n = \boxed{\frac{f^{(n)}(x_0)}{n!}}$$

**b.** In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) = \boxed{f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N}$$

**c.** In closed form (i.e., with a $\sum$-sign and without ... )

$$P_N(x) = \boxed{\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n}$$

**d.** In open form (i.e., with ... and without a $\sum$-sign)

$$P_\infty(x) = \boxed{f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots}$$

**e.** In closed form (i.e., with a $\sum$-sign and without ... )

$$P_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n}$$

**f.** We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \boxed{\frac{f^{(N+1)}(c)}{(N+1)!}\,(x - x_0)^{(N+1)}} \text{ for some } c \text{ between } \boxed{x} \text{ and } \boxed{x_0}.$$

**g.** A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = \boxed{0}$.

## Commonly Used Taylor Series

▶. Here, *expansion* refers to the power series expansion that is the Maclaurin series.

- An expansion for $y = e^x$ is $\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

- An expansion for $y = \cos x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

- An expansion for $y = \sin x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

- An expansion for $y = \frac{1}{1-x}$ is $\boxed{\sum_{n=0}^{\infty} x^n}$, which is valid precisely when $x \in \boxed{(-1, 1)}$.

- An expansion for $y = \ln(1+x)$ is $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$, which is valid precisely when $x \in \boxed{(-1, 1]}$.

- An expansion for $y = \arctan x$ is $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$, which is valid precisely when $x \in \boxed{[-1, 1]}$.

## Polar Coordinates

▶. Here, CC stands for *Cartresian coordinates* while PC stands for *polar coordinates.*

- A point with PC $(r, \theta)$ also has PC $\left(\boxed{r}, \theta + 2\pi\right)$ as well as $\left(\boxed{-r}, \theta + \pi\right)$.
- A point $P \in \mathbb{R}^2$ with CC $(x, y)$ and PC $(r, \theta)$ satisfies the following.

$$x = \boxed{r\ \cos\theta} \quad \& \quad y = \boxed{r\ \sin\theta} \quad \& \quad r^2 = \boxed{x^2 + y^2} \quad \& \quad \boxed{\tan\theta} = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases}.$$

- The period of $f(\theta) = \cos(k\theta)$ and s of $f(\theta) = \sin(k\theta)$ is $\boxed{\frac{2\pi}{k}}$ To sketch these graphs, we divide the period by $\boxed{4}$ and make *the chart*, in order to detect the $\boxed{\text{max/min/zero's of the function } r = f(\theta)}$.
- Now consider a function $r = f(\theta)$ which determines a curve in the plane where
  (1) $f \colon [\alpha, \beta] \to [0, \infty]$
  (2) $f$ is continuous on $[\alpha, \beta]$
  (3) $\beta - \alpha \leq 2\pi$ .

  Then the area bounded by polar curves $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\theta=\alpha}^{\theta=\beta} \boxed{\frac{1}{2}\,[f(\theta)]^2}\ d\theta\ .$$

**Area and Volume of Revolutions**

Let's start with some region $R$ in the (2 dimensional) $xy$-plane and revolve $R$ around an axis of revolution to generate a (3 dimensional) solid of revolution $S$. Next we want to find the area of $R$ as well as the volume of $S$.

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- **In parts a, fill in the boxes with:** $x$ **or** $y$**.**
- **In parts b, c, and d, fill in the boxes with a formula involving *some* of:** **2 , $\pi$ , radius , base , radius$_{\text{big}}$ , radius$_{\text{little}}$ , average radius , height , and/or thickness .**

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▶. **Area via Riemann Sums**. Let's find the area of $R$ by forming typical rectangles.

**a.** We first partition either the [ $x$ ]-axis or the [ $y$ ]-axis. (We can pick either.)

⊡. Next, using the partition, we form typical rectangles. Then we find the area of each typical rectangle.

**b.** If we partition the $z$-axis, where $z$ is either $x$ or $y$, the $\Delta z =$ [ base ] of a typical rectangle.

**c.** The area of a typical rectangle is [ (height) (base) ].

▶. **Disk/Washer Method**. Let's find the volume of the solid of revolution $S$ using the disk/washer method.

**a.** If the axis of revolution is:

- the $x$-axis, or parallel to the $x$-axis, then we partition the [ $x$ ]-axis.
- the $y$-axis, or parallel to the $y$-axis, then we partition the [ $y$ ]-axis.

⊡. Next, using the partition, we form typical disk/washer's. Then we find the volume of each typical disk/washer.

**b.** If we partition the $z$-axis, where $z$ is either $x$ or $y$, the $\Delta z =$ [ height ] of a tyical disk/washer.

**c.** If we use the **disk method**, then the volume of a typical disk is:

$$\pi \ (\text{radius})^2 \ (\text{height})$$ .

**d.** If we use the **washer method**, then the volume of a typical washer is: (either form is fine)

$$\pi \ (\text{radius}_{\text{big}})^2 \ (\text{height}) \ - \ \pi \ (\text{radius}_{\text{little}})^2 \ (\text{height}) \ \stackrel{\text{or}}{=} \ \pi \ \left[(\text{radius}_{\text{big}})^2 - (\text{radius}_{\text{little}})^2\right] \ (\text{height})$$ .

▶. **Shell Method**. Let's find the volume of this solid of revolution $S$ using the shell method.

**a.** If the axis of revolution is:

- the $x$-axis, or parallel to the $x$-axis, then we partition the [ $y$ ]-axis.
- the $y$-axis, or parallel to the $y$-axis, then we partition the [ $x$ ]-axis.

⊡. Next, using the partition, we form typical shells. Then we find the volume of each typical shell.

**b.** If we partition the $z$-axis, where $z$ is either $x$ or $y$, the $\Delta z =$ [ thickness ] of a typical shell.
Also acceptable is $\Delta z \ = \ \text{radius}_{\text{big}} \ - \ \text{radius}_{\text{little}}$.

**c.** The volume of a typical shell is: (either form is fine)

$$2\pi \ (\text{average radius}) \ (\text{height}) \ (\text{thickness}) \ \stackrel{\text{or}}{=} \ 2\pi \ (\text{radius}) \ (\text{height}) \ (\text{thickness})$$