Formula/Concepts You Need To Know

Review of some needed Trig. Identities for Integration

- Your answers should be an angle in **RADIANS**.
  - $\arccos(\ \frac{1}{2}\ ) =$ __________
  - $\arccos(\ -\frac{1}{2}\ ) =$ __________
  - $\arcsin(\frac{1}{2}) =$ __________
  - $\arcsin(\ -\frac{1}{2}\ ) =$ __________
  - Can you do similar problems?
- Double-angle formulas. Your answer should involve trig functions of θ, and not of 2θ.
  - $\cos(2\theta) =$ ________________
  - $\sin(2\theta) =$ ________________ .
- Half-angle formulas. Your answer should involve $\cos(2\theta)$.
  - $\cos^2(\theta) =$ [ ]
  - $\sin^2(\theta) =$ [ ]
- Since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween:
  - tangent (i.e., tan) and secant (i.e., sec) is __________________________ .
  - cotangent (i.e., cot) and cosecant (i.e., csc) is __________________________ .

Remember Your Calculus I Integration Basics?

In this part, a is a constant and $a > 0$.

- If $u \neq 0$, then $\int \frac{du}{u} =$ __________________________ + C
- If $a \neq 1$, then $\int a^u\, du =$ __________________________ + C
- $\int \cos u\, du =$ ____ $+ C$
- $\int \sec^2 u\, du =$ ____ $+ C$
- $\int \sec u \tan u\, du =$ ____ $+ C$
- $\int \sin u\, du =$ ____ $+ C$
- $\int \csc^2 u\, du =$ ____ $+ C$
- $\int \csc u \cot u\, du =$ ____ $+ C$
- $\int \tan u\, du =$ ____ $+ C$
- $\int \cot u\, du =$ ____ $+ C$
- $\int \sec u\, du =$ ____ $+ C$
- $\int \csc u\, du =$ ____ $+ C$
- $\int \frac{1}{\sqrt{a^2-u^2}}\, du =$ __________________________ $+ C$
- $\int \frac{1}{a^2+u^2}\, du =$ __________________________ $+ C$
- $\int \frac{1}{u\sqrt{u^2-a^2}}\, du =$ __________________________ $+ C$

Integration from Calculus II

- Partial Fraction Decomposition. Let's integrate $y = \frac{f(x)}{g(x)}$, where f and g are polynomials, by 1[st] finding its PDF.
  - If [degree of f] $\geq$ [degree of g], then one must first do ______________________________ .
  - If [degree of f] $<$ [degree of g], then first factor $y = g(x)$ into [ ] factors $px + q$ and irreducible [ ] factors $ax^2 + bx + c$ (to be sure it's irreducible, you need [ ]) .

  Next, collect up like terms and follow the following rules.

Rule 1: For each factor of the form $(px+q)^m$ where $m \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of [ ] partial fractions of the form, where each A_i is a real number,

[ ] .

Rule 2: For each factor of the form $(ax^2 + bx + c)^n$ where $n \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of [ ] partial fractions of the form, where the A_i's and B_i's are real number,

[ ] .

- Integration by parts formula: $\int u\,dv =$ ______________________________
- Trig. Substitution. (Recall that the *integrand* is the function you are integrating.) Here, a is a constant and $a > 0$.
  - if the integrand involves $a^2 - u^2$, then one makes the substitution $u =$ ____________ .
  - if the integrand involves $a^2 + u^2$, then one makes the substitution $u =$ ____________ .
  - if the integrand involves $u^2 - a^2$, then one makes the substitution $u =$ ____________ .

Sequences

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Complete the below sentences.
  - The limit of $\{a_n\}_{n=1}^{\infty}$ is the real number L provided for each $\epsilon > 0$ there exists a natural number N so that if the natural number n satisfies ____ $>$ ____ then ________ $<$ ____ .
  - If the limit of $\{a_n\}_{n=1}^{\infty}$ is $L \in \mathbb{R}$, then we denote this by ____________ .
  - $\{a_n\}_{n=1}^{\infty}$ converges provided ______________________________ .
  - $\{a_n\}_{n=1}^{\infty}$ diverges provided $\{a_n\}_{n=1}^{\infty}$ ______________________________ .
  - $\{a_n\}_{n=1}^{\infty}$ diverges provided $\lim_{n\to\infty} a_n$ ______________________________ .
- Practice taking basic limits. (Important, e.g., for Ratio and Root Tests.)
  - $\lim\limits_{n\to\infty} \dfrac{5n^{17} + 6n^2 + 1}{7n^{18} + 9n^3 + 5} =$ __________
  - $\lim\limits_{n\to\infty} \dfrac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5} =$ __________
  - $\lim\limits_{n\to\infty} \dfrac{-5n^{18} + 6n^2 + 1}{7n^{17} + 9n^3 + 5} =$ __________
  - $\lim\limits_{n\to\infty} \sqrt{\dfrac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5}} =$ __________
  - Can you do similar problems?
- Let $-\infty < r < \infty$. (Needed for Geometric Series. Warning, don't confuse sequences with series.)
  - If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r = 1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r > 1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r = -1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r < -1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .

Series

►. In this section, all series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

•. For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums, so $s_N = S\sum_{n=1}^{N} a_n$. Then the formal series $\sum a_n$

- converges if and only if ______________________________
- converges to $L \in \mathbb{R}$ if and only if ______________________________
- diverges if and only if ______________________________ .

Now assume, furthermore, that $a_n \geq 0$ for each n. Then the sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums either

- is bounded above (by some finite number), in which case the series $\sum a_n$ __________

or

- is not bounded above (by some finite number), in which case the series $\sum a_n$ __________ .

•. The n^{th}**-term test** for an arbitrary series $\sum a_n$.
If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ [ ] .

•. Fix $r \in \mathbb{R}$. For $N \geq 17$, let $s_N = \sum_{\mathbf{n=17}}^{N} r^n$ (Note the sum starts at 17). Then, for $N > 17$,

- $s_N =$ ______________________________ (your answer can have . . .'s but not a $\sum$ sign)
- $r\, s_N =$ ______________________________ (your answer can have . . .'s but not a $\sum$ sign)
- $(1 - r)\, s_N =$ ______________________ (your answer should have neither . . .'s nor a $\sum$ sign)
- and if $r \neq 1$, then $s_N =$ ______________ (your answer should have neither . . .'s nor a $\sum$ sign)

•. **Geometric Series** where $-\infty < r < \infty$. The series $\sum r^n$ (hint: look at the previous questions):

- converges if and only if $|r|$ [ ]
- diverges if and only if $|r|$ [ ] .

•. **Integral Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$. Let $f\colon [1, \infty) \to \mathbb{R}$ be so that $a_n = f(\;[\;]\;)$ for each $n \in \mathbb{N}$ and $y = f(x)$ is a [ ] , [ ] , [ ] function. Then we have the following.

(1) For each $N > 2$,

$$\sum_{n=\Box}^{\Box} a_n \;\leq\; \int_{x=1}^{x=N} f(x)\,dx \;\leq\; \sum_{n=\Box}^{\Box} a_n\,. \tag{1}$$

Fill in, as so to give the best estimate one can, each of the 4 boxes with: a number, N, $N - 1$, or $N + 1$.

Hint. Approximate (below and above) the $\int_1^N f(x)\,dx$ by the area of $N - 1$ Riemann rectangles, each of base length $\Delta x = 1$.

(2) From the bounds in (1), we see that $\sum a_n$ converges if and only if [ ] converges.

(3) Now let $\sum a_n$ converge. We want to approximate the infinite sum $\sum_{n=1}^{\infty} a_n$ by the finite sum $\sum_{n=1}^{N} a_n$ within an error (i.e., remainder) of R_N. To figure out how good this approximation is, define R_N as below and get a good (as one can) lower and upper approximation of R_N, again using Reimann sums. Fill in the 3 boxes with: a number, N, $N-1$, or $N+1$.

$$\boxed{} \le R_N \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=\boxed{}}^{\infty} a_n \le \int_{x=\boxed{}}^{\infty} f(x)\,dx\ .$$

- **p-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$
  - converges if and only if p $\boxed{}$.
  - diverges if and only if p $\boxed{}$.

  This can be shown by using the ________ test and comparing $\sum \frac{1}{n^p}$ to (the easy to compute) $\int_{x=1}^{\infty}$ ________ dx .
- **Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \ge 0$. (Fill in the blanks with a_n and/or b_n.)
  - If $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$ and $\sum$ $\boxed{}$ converge, then $\sum$ $\boxed{}$ converge.
  - If $0 \le b_n \le a_n$ for all $n \in \mathbb{N}$ and $\sum$ $\boxed{}$ diverge, then $\sum$ $\boxed{}$ diverge.

  Hint: sing the song to yourself.
- **Limit Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \ge 0$. Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.
  If $\boxed{} < L < \boxed{}$, then $\sum a_n$ converges if and only if $\boxed{}$.
- **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}\ .$$

  - If ρ $\boxed{}$ then $\sum a_n$ converges absolutely.
  - If ρ $\boxed{}$ then $\sum a_n$ diverges.
  - If ρ $\boxed{}$ then the test is inconclusive (in other words, the test fails).
- **Alternating Series Test (AST) & Alternating Series Estimation Theorem (ASET).**
  Consider an alternating series $\sum(-1)^n u_n$ where $u_n > 0$ for each $n \in \mathbb{N}$.
  If
  - u_n $\boxed{}$ u_{n+1} for each $n \in \mathbb{N}$
  - $\lim_{n\to\infty} u_n =$ $\boxed{}$

  then
  - $\sum(-1)^n u_n$ $\boxed{}$
  - we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty}(-1)^n u_n$ by the finite sum $\sum_{n=1}^{N}(-1)^n u_n$ and the error (i.e. remainder) satisfies

$$\left| \sum_{n=1}^{\infty}(-1)^n u_n - \sum_{n=1}^{N}(-1)^n u_n \right| \le \boxed{}\ .$$

- By definition, for an arbitrary series $\sum a_n$, (fill in these 4 boxes with converges or diverges).
  - $\sum a_n$ is <u>absolutely convergent</u> if and only if $\sum |a_n|$ $\boxed{}$.
  - $\sum a_n$ is <u>conditionally convergent</u> if and only if $\sum a_n$ $\boxed{}$ and $\sum |a_n|$ $\boxed{}$.

- $\sum a_n$ is divergent if and only if $\sum a_n$ ______ .

•. Fill in the 3 blank boxes with absolutely convergent, conditionally convergent, or divergent) on the following FLOW CHART from class used to determine the behavior of a series $\sum_{n=17}^{\infty} a_n$.

Does $\sum |a_n|$ converge? Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$ Does $\lim_{n\to\infty} |a_n| = 0$? if NO $\Rightarrow$ $\sum a_n$ is ______

if YES $\Downarrow$ (from the first box): $\sum a_n$ is ______

if YES $\Downarrow$ (from "Does $\lim_{n\to\infty} |a_n| = 0$?"): Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is ______

Power Series

Condsider a (formal) power series

$$h(x) \;=\; \sum_{n=0}^{\infty} a_n \, (x - x_0)^n \;, \tag{2}$$

with radius of convergence $R \in [0, \infty]$. (Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

•. Fill in the next for boxes with one of the following 4 choices: [**a.**] is always absolutely convergent (AC) [**b.**] is always conditionally convergent (CC) [**c.**] is always divergent (DIV) [**d.**] can do anything, i.e., there are examples showing that it can be AC, CC, or DIV.

(1) For $x = x_0$, the power series $h(x)$ in (2) ______ .

(2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series $h(x)$ in (2) ______ .

(3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$ the power series $h(x)$ in (2) ______ .

(4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series $h(x)$ in (2) ______ .

•. For the next 2 problems, let $R > 0$ and fill-in the boxes. Consider the function $y = h(x)$ defined by the power series in (2).

(1) The function $y = h(x)$ is always differentiable on the interval ______ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

$$h'(x) \;=\; \sum_{n=1}^{\infty} \boxed{} \;. \tag{3}$$

What can you say about the radius of convergence of the power series in (3)? ______.

(2) The function $y = h(x)$ always has an antiderivative on the interval ______ (make this interval as large as it can be, but still keeping the statement true). Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\, dx \;=\; \sum_{n=0}^{\infty} \boxed{} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

Taylor/Maclaurin Polynomials and Series

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. The formula for c_n is

$c_n =$ [ ]

b. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$P_N(x) =$ [ ]

c. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$P_N(x) =$ [ ]

d. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$P_\infty(x) =$ [ ]

e. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$P_\infty(x) =$ [ ]

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$R_N(x) =$ [ ] for some c between [ ] and [ ].

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 =$ [ ].

Commonly Used Taylor Series

►. Here, *expansion* refers to the power series expansion that is the Maclaurin series.

- An expansion for $y = e^x$ is $\boxed{\quad\quad\quad}$, which is valid precisely when $x \in \boxed{\quad\quad}$.

- An expansion for $y = \cos x$ is $\boxed{\quad\quad\quad}$, which is valid precisely when $x \in \boxed{\quad\quad}$.

- An expansion for $y = \sin x$ is $\boxed{\quad\quad\quad}$, which is valid precisely when $x \in \boxed{\quad\quad}$.

- An expansion for $y = \frac{1}{1-x}$ is $\boxed{\quad\quad\quad}$, which is valid precisely when $x \in \boxed{\quad\quad}$.

- An expansion for $y = \ln(1+x)$ is $\boxed{\quad\quad\quad}$, which is valid precisely when $x \in \boxed{\quad\quad}$.

- An expansion for $y = \arctan x$ is $\boxed{\quad\quad\quad}$, which is valid precisely when $x \in \boxed{\quad\quad}$.

Polar Coordinates

►. Here, CC stands for *Cartresian coordinates* while PC stands for *polar coordinates*.

- A point with PC (r, θ) also has PC $\left(\boxed{\quad}\ ,\ \theta + 2\pi\right)$ as well as $\left(\boxed{\quad}\ ,\ \theta + \pi\right)$.
- A point $P \in \mathbb{R}^2$ with CC (x, y) and PC (r, θ) satisfies the following.

$$x = \boxed{\quad\quad} \quad \& \quad y = \boxed{\quad\quad} \quad \& \quad r^2 = \boxed{\quad\quad} \quad \& \quad \boxed{\quad\quad} = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases} .$$

- The period of $f(\theta) = \cos(k\theta)$ and s of $f(\theta) = \sin(k\theta)$ is $\boxed{\quad}$ To sketch these graphs, we divide the period by $\boxed{\quad}$ and make *the chart*, in order to detect the $\boxed{\quad\quad\quad\quad\quad\quad\quad\quad}$.
- Now consider a function $r = f(\theta)$ which determines a curve in the plane where
  (1) $f\colon [\alpha, \beta] \ \to \ [0, \infty]$
  (2) f is continuous on $[\alpha, \beta]$
  (3) $\beta - \alpha \leq 2\pi$.

  Then the area bounded by polar curves $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\theta=\alpha}^{\theta=\beta} \boxed{\quad\quad\quad\quad} \, d\theta .$$

Area and Volume of Revolutions

Let's start with some region R in the (2 dimensional) xy-plane and revolve R around an axis of revolution to generate a (3 dimensional) solid of revolution S. Next we want to find the area of R as well as the volume of S.

- **In parts a, fill in the boxes with:** x **or** y**.**
- **In parts b, c, and d, fill in the boxes with a formula involving *some* of:** **2 , π , radius , base , radius$_{\text{big}}$, radius$_{\text{little}}$, average radius , height , and/or thickness** .

▶. **Area via Riemann Sums**. Let's find the area of R by forming typical rectangles.

a. We first partition either the [ ]–axis or the [ ]–axis.

⊡. Next, using the partition, we form typical rectangles. Then we find the area of each typical rectangle.

b. If we partition the z-axis, where z is either x or y, the $\Delta z =$ [ ] of a typical rectangle.

c. The area of a typical rectangle is [ ].

▶. **Disk/Washer Method**. Let's find the volume of the solid of revolution S using the disk/washer method.

a. If the axis of revolution is:

- the x-axis, or parallel to the x-axis, then we partition the [ ]–axis.
- the y-axis, or parallel to the y-axis, then we partition the [ ]–axis.

⊡. Next, using the partition, we form typical disk/washer's. Then we find the volume of each typical disk/washer.

b. If we partition the z-axis, where z is either x or y, the $\Delta z =$ [ ] of a tyical disk/washer.

c. If we use the **disk method**, then the volume of a typical disk is:

[ ] .

d. If we use the **washer method**, then the volume of a typical washer is:

[ ] .

▶. **Shell Method**. Let's find the volume of this solid of revolution S using the shell method.

a. If the axis of revolution is:

- the x-axis, or parallel to the x-axis, then we partition the [ ]–axis.
- the y-axis, or parallel to the y-axis, then we partition the [ ]–axis.

⊡. Next, using the partition, we form typical shells. Then we find the volume of each typical shell.

b. If we partition the z-axis, where z is either x or y, the $\Delta z =$ [ ] of a typical shell.

c. The volume of a typical shell is:

[ ]