Formula/Concepts You Need To Know

Review of some needed Trig. Identities for Integration

• Your answers should be an angle in RADIANS.
  - \( \arccos\left(\frac{1}{2}\right) = \) ________  \( \arccos\left(-\frac{1}{2}\right) = \) ________
  - \( \arcsin\left(\frac{1}{2}\right) = \) ________  \( \arcsin\left(-\frac{1}{2}\right) = \) ________
  - Can you do similar problems?

• Double-angle formulas. Your answer should involve trig functions of \( \theta \), and not of \( 2\theta \).
  - \( \cos(2\theta) = \) ____________  \( \sin(2\theta) = \) ____________

• Half-angle formulas. Your answer should involve \( \cos(2\theta) \).
  - \( \cos^2(\theta) = \) ________  \( \sin^2(\theta) = \) ________

• Since \( \cos^2\theta + \sin^2\theta = 1 \), we know that the corresponding relationship between:
  - tangent (i.e., tan) and secant (i.e., sec) is ________________.
  - cotangent (i.e., cot) and cosecant (i.e., csc) is ________________.

Remember Your Calculus I Integration Basics?  In this part, \( a \) is a constant and \( a > 0 \).

• If \( u \neq 0 \), then \( \int \frac{du}{u} = \) ____________ + C
• If \( a \neq 1 \), then \( \int a^u \, du = \) ____________ + C
• \( \int \cos u \, du = \) ____________ + C
• \( \int \sec^2 u \, du = \) ____________ + C
• \( \int \sec u \tan u \, du = \) ____________ + C
• \( \int \sin u \, du = \) ____________ + C
• \( \int \csc^2 u \, du = \) ____________ + C
• \( \int \csc u \cot u \, du = \) ____________ + C
• \( \int \tan u \, du = \) ____________ + C
• \( \int \cot u \, du = \) ____________ + C
• \( \int \sec u \, du = \) ____________ + C
• \( \int \csc u \, du = \) ____________ + C
• \( \int \frac{1}{\sqrt{a^2-u^2}} \, du = \) ____________ + C
• \( \int \frac{1}{a^2+u^2} \, du = \) ____________ + C
• \( \int \frac{1}{u\sqrt{u^2-a^2}} \, du = \) ____________ + C
Integration from Calculus II

- **Partial Fraction Decomposition.** Let’s integrate \( y = \frac{f(x)}{g(x)} \), where \( f \) and \( g \) are polynomials, by 1st finding its PDF.
  - If [degree of \( f \)] \( \geq \) [degree of \( g \)], then one must first do __________.
  - If [degree of \( f \)] \( < \) [degree of \( g \)], then first factor \( y = g(x) \) into factors \( px + q \) and irreducible factors \( ax^2 + bx + c \) (to be sure it’s irreducible, you need __________) .

**Rule 1:** For each factor of the form \((px + q)^m\) where \( m \geq 1 \), the decomposition of \( y = \frac{f(x)}{g(x)} \) contains a sum of partial fractions of the form, where each \( A_i \) is a real number,

**Rule 2:** For each factor of the form \((ax^2 + bx + c)^n\) where \( n \geq 1 \), the decomposition of \( y = \frac{f(x)}{g(x)} \) contains a sum of partial fractions of the form, where the \( A_i \)'s and \( B_i \)'s are real number,

- **Integration by parts formula:** \( \int u \, dv = \ldots \)
- **Trig. Substitution.** (Recall that the integrand is the function you are integrating.) Here, \( a \) is a constant and \( a > 0 \).
  - if the integrand involves \( a^2 - u^2 \), then one makes the substitution \( u = \ldots \).
  - if the integrand involves \( a^2 + u^2 \), then one makes the substitution \( u = \ldots \).
  - if the integrand involves \( u^2 - a^2 \), then one makes the substitution \( u = \ldots \).

- **Sequences**
  - Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of real numbers. Complete the below sentences.
    - The limit of \( \{a_n\}_{n=1}^{\infty} \) is the real number \( L \) provided for each \( \epsilon > 0 \) there exists a natural number \( N \) so that if the natural number \( n \) satisfies ___ > ___ then ___ < ___.
    - If the limit of \( \{a_n\}_{n=1}^{\infty} \) is \( L \in \mathbb{R} \), then we denote this by __________.
    - \( \{a_n\}_{n=1}^{\infty} \) converges provided __________.
    - \( \{a_n\}_{n=1}^{\infty} \) diverges provided \( \{a_n\}_{n=1}^{\infty} \) __________.
    - \( \{a_n\}_{n=1}^{\infty} \) diverges provided \( \lim_{n \to \infty} a_n \) __________.
  - **Practice taking basic limits.** (Important, e.g., for Ratio and Root Tests.)
    - \( \lim_{n \to \infty} \frac{5n^{17} + 6n^2 + 1}{7n^{18} + 9n^3 + 5} = \ldots \)
    - \( \lim_{n \to \infty} \frac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5} = \ldots \)
    - \( \lim_{n \to \infty} \frac{-5n^{18} + 6n^2 + 1}{7n^{17} + 9n^3 + 5} = \ldots \)
    - \( \lim_{n \to \infty} \sqrt[3]{\frac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5}} = \ldots \)
  - Can you do similar problems?
  - Let \( -\infty < r < \infty \). (Needed for Geometric Series. Warning, don’t confuse sequences with series.)
    - If \( |r| < 1 \), then \( \lim_{n \to \infty} r^n = \ldots \).
    - If \( r = 1 \), then \( \lim_{n \to \infty} r^n = \ldots \).
    - If \( r > 1 \), then \( \lim_{n \to \infty} r^n = \ldots \).
    - If \( r = -1 \), then \( \lim_{n \to \infty} r^n = \ldots \).
    - If \( r < -1 \), then \( \lim_{n \to \infty} r^n = \ldots \).
In this section, all series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

- For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums, so $s_N = \sum_{n=1}^{N} a_n$. Then the formal series $\sum a_n$
  - converges if and only if ______________________
  - converges to $L \in \mathbb{R}$ if and only if ______________________
  - diverges if and only if ______________________.

Now assume, furthermore, that $a_n \geq 0$ for each $n$. Then the sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums either
  - is bounded above (by some finite number), in which case the series $\sum a_n$ ________________
  - or
  - is not bounded above (by some finite number), in which case the series $\sum a_n$ ________________.

- The $n$th-term test for an arbitrary series $\sum a_n$.
  If $\lim_{n \to \infty} a_n \neq 0$ or $\lim_{n \to \infty} a_n$ does not exist, then $\sum a_n$ ________________.

- Fix $r \in \mathbb{R}$. For $N \geq 17$, let $s_N = \sum_{n=17}^{N} r^n$ (Note the sum starts at 17). Then, for $N > 17$,
  - $s_N =$ ______________________ (your answer can have \ldots’s but not a $\sum$ sign)
  - $r s_N =$ ______________________ (your answer can have \ldots’s but not a $\sum$ sign)
  - $(1 - r) s_N =$ ______________________ (your answer should have neither \ldots’s nor a $\sum$ sign)
  - and if $r \neq 1$, then $s_N =$ ______________________ (your answer should have neither \ldots’s nor a $\sum$ sign)

- Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$ (hint: look at the previous questions):
  - converges if and only if $|r|$ ________________
  - diverges if and only if $|r|$ ________________.

- Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$. Let $f: [1, \infty) \to \mathbb{R}$ be so that $a_n = f(\square)$ for each $n \in \mathbb{N}$ and $y = f(x)$ is a __________ function, then we have the following.
  (1) For each $N > 2$,
    $$\sum_{n=1}^{N} a_n \leq \int_{x=1}^{x=N} f(x) \, dx \leq \sum_{n=\square}^{\square} a_n \, .$$
    (1)
    Fill in, as so to give the best estimate one can, each of the 4 boxes with: a number, $N$, $N - 1$, or $N + 1$.
    Hint. Approximate (below and above) the $\int_{1}^{N} f(x) \, dx$ by the area of $N - 1$ Riemann rectangles, each of base length $\Delta x = 1$.
  (2) From the bounds in (1), we see that $\sum a_n$ converges if and only if ________________ converges.
(3) Now let $\sum a_n$ converge. We want to approximate the infinite sum $\sum_{n=1}^{\infty} a_n$ by the finite sum $\sum_{n=1}^{N} a_n$ within an error (i.e., remainder) of $R_N$. To figure out how good this approximation is, define $R_N$ as below and get a good (as one can) lower and upper approximation of $R_N$, again using Reimann sums. Fill in the 3 boxes with: a number, $N$, $N-1$, or $N+1$.

$$ R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=1}^{\infty} a_n - \int_{x=1}^{\infty} f(x) \, dx. $$

- **p-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$
  - converges if and only if $p > 1$.
  - diverges if and only if $p \leq 1$.

This can be shown by using the **comparison test** and comparing $\sum \frac{1}{n^p}$ to (the easy to compute) $\int_{x=1}^{\infty} \frac{1}{x} \, dx$.

- **Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$.
  - If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converge, then $\sum a_n$ converge.
  - If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverge, then $\sum a_n$ diverge.

Hint: sing the song to yourself.

- **Limit Comparison Test** for a positive-termed series $\sum a_n$ where $a_n \geq 0$. Let $b_n > 0$ and $\lim_{n \to \infty} \frac{a_n}{b_n} = L$.
  - If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converge.
  - If $L < 0$, then $\sum a_n$ diverges.
  - If $L > 0$, then the test is inconclusive (in other words, the test fails).

- **Alternating Series Test (AST) & Alternating Series Estimation Theorem (ASET).**
  Consider an alternating series $\sum (-1)^n u_n$ where $u_n > 0$ for each $n \in \mathbb{N}$.
  If
  - $u_n$ $u_{n+1}$ for each $n \in \mathbb{N}$
  - $\lim_{n \to \infty} u_n$ =
  then
  - $\sum (-1)^n u_n$
  - we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty} (-1)^n u_n$ by the finite sum $\sum_{n=1}^{N} (-1)^n u_n$ and the error (i.e. remainder) satisfies
    $$ \left| \sum_{n=1}^{\infty} (-1)^n u_n - \sum_{n=1}^{N} (-1)^n u_n \right| \leq \square. $$
  - By definition, for an arbitrary series $\sum a_n$, (fill in these 4 boxes with converges or diverges).
    - $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$
    - $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ and $\sum |a_n|$. 


• \( \sum a_n \) is divergent if and only if \( \sum a_n \) .

• Fill in the 3 blank boxes with absolutely convergent, conditionally convergent, or divergent) on the following FLOW CHART from class used to determine the behavior of a series \( \sum_{n=1}^{\infty} a_n \).

| Does \( \sum |a_n| \) converge? |
|----------------------------------|
| Yes                               |
| \( \sum a_n \) is               |
| if NO                             |
|                              \[
\lim_{n \to \infty} |a_n| = 0 \]
| if NO \( \sum a_n \) is         |
| Is \( \sum a_n \) an alternating series? |
| if YES \( \sum a_n \) is       |
| Does \( \sum a_n \) satisfy the conditions of the Alternating Series Test? |
| if YES \( \sum a_n \) is       |

**Power Series**

Consider a (formal) power series

\[
h(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{2}
\]

with radius of convergence \( R \in [0, \infty] \). (Here \( x_0 \in \mathbb{R} \) is fixed and \( \{a_n\}_{n=0}^{\infty} \) is a fixed sequence of real numbers.)

• Fill in the next 4 boxes with one of the following 4 choices: 
  - [a.] is always absolutely convergent (AC)
  - [b.] is always conditionally convergent (CC)
  - [c.] is always divergent (DIV)
  - [d.] can do anything, i.e., there are examples showing that it can be AC, CC, or DIV.

  1. For \( x = x_0 \), the power series \( h(x) \) in (2) ______.
  2. For \( x \in \mathbb{R} \) such that \( |x - x_0| < R \), the power series \( h(x) \) in (2) ______.
  3. For \( x \in \mathbb{R} \) such that \( |x - x_0| > R \) the power series \( h(x) \) in (2) ______.
  4. If \( R > 0 \), then for the endpoints \( x = x_0 \pm R \), the power series \( h(x) \) in (2) ______.

• For the next 2 problems, let \( R > 0 \) and fill-in the boxes. Consider the function \( y = h(x) \) defined by the power series in (2).

  1. The function \( y = h(x) \) is always differentiable on the interval ______ (make this interval as large as it can be, but still keeping the statement true). Furthermore, on this interval

\[
h'(x) = \sum_{n=1}^{\infty} \quad \tag{3}
\]

What can you say about the radius of convergence of the power series in (3)? ______

  2. The function \( y = h(x) \) always has an antiderivative on the interval ______ (make this interval as large as it can be, but still keeping the statement true). Furthermore, if \( \alpha \) and \( \beta \) are in this interval, then

\[
\int_{x=\alpha}^{x=\beta} h(x) \, dx = \sum_{n=0}^{\infty} \quad |x=\beta| \, x=\alpha.
\]
Taylor/Maclaurin Polynomials and Series

Let \( y = f(x) \) be a function with derivatives of all orders in an interval \( I \) containing \( x_0 \).

Let \( y = P_N(x) \) be the \( N \)-th-order Taylor polynomial of \( y = f(x) \) about \( x_0 \).

Let \( y = R_N(x) \) be the \( N \)-th-order Taylor remainder of \( y = f(x) \) about \( x_0 \).

Let \( y = P_\infty(x) \) be the Taylor series of \( y = f(x) \) about \( x_0 \).

Let \( c_n \) be the \( n \)-th Taylor coefficient of \( y = f(x) \) about \( x_0 \).

a. The formula for \( c_n \) is

\[ c_n = \]

b. In open form (i.e., with \( ... \) and without a \( \sum \)-sign)

\[ P_N(x) = \]

c. In closed form (i.e., with a \( \sum \)-sign and without \( ... \))

\[ P_N(x) = \]

d. In open form (i.e., with \( ... \) and without a \( \sum \)-sign)

\[ P_\infty(x) = \]

e. In closed form (i.e., with a \( \sum \)-sign and without \( ... \))

\[ P_\infty(x) = \]

f. We know that \( f(x) = P_N(x) + R_N(x) \). Taylor’s BIG Theorem tells us that, for each \( x \in I \),

\[ R_N(x) = \]

for some \( c \) between \( \square \) and \( \square \).

\[ f(x) = P_N(x) + R_N(x) \]

\[ \text{for some } c \text{ between } \square \text{ and } \square \].

g. A Maclaurin series is a Taylor series with the center specifically specified as \( x_0 = \)

\[ \square \].
Commonly Used Taylor Series

- Here, \textit{expansion} refers to the power series expansion that is the Maclaurin series.
  - An expansion for \( y = e^x \) is , which is valid precisely when \( x \in \) .
  - An expansion for \( y = \cos x \) is , which is valid precisely when \( x \in \) .
  - An expansion for \( y = \sin x \) is , which is valid precisely when \( x \in \) .
  - An expansion for \( y = \frac{1}{1-x} \) is , which is valid precisely when \( x \in \) .
  - An expansion for \( y = \ln(1+x) \) is , which is valid precisely when \( x \in \) .
  - An expansion for \( y = \arctan x \) is , which is valid precisely when \( x \in \) .

Polar Coordinates

- Here, CC stands for \textit{Cartesian coordinates} while PC stands for \textit{polar coordinates}.
  - A point with PC \((r, \theta)\) also has PC \((r, \theta + 2\pi)\) as well as \((r, \theta + \pi)\).
  - A point \( P \in \mathbb{R}^2 \) with CC \((x, y)\) and PC \((r, \theta)\) satisfies the following.
    \[
    x = \square \quad \& \quad y = \square \quad \& \quad r^2 = \square \quad \& \quad \square = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases}.
    \]
  - The period of \( f(\theta) = \cos(k\theta) \) and \( s \) of \( f(\theta) = \sin(k\theta) \) is \square. To sketch these graphs, we divide the period by \square and make the chart, in order to detect the \square.
  - Now consider a function \( r = f(\theta) \) which determines a curve in the plane where
    \begin{enumerate}
    \item \( f : [\alpha, \beta] \rightarrow [0, \infty] \)
    \item \( f \) is continuous on \([\alpha, \beta]\)
    \item \( \beta - \alpha \leq 2\pi \).
    \end{enumerate}
  Then the area bounded by polar curves \( r = f(\theta) \) and the rays \( \theta = \alpha \) and \( \theta = \beta \) is
  \[
  A = \int_{\theta=\alpha}^{\theta=\beta} \square \, d\theta.
  \]
Area and Volume of Revolutions

Let’s start with some region $R$ in the (2 dimensional) $xy$-plane and revolve $R$ around an axis of revolution to generate a (3 dimensional) solid of revolution $S$. Next we want to find the area of $R$ as well as the volume of $S$.

- In parts a, fill in the boxes with: $x$ or $y$.
- In parts b, c, and d, fill in the boxes with a formula involving some of: $2$, $\pi$, radius, base, radius_{big}, radius_{little}, average radius, height, and/or thickness.

▶ Area via Riemann Sums. Let’s find the area of $R$ by forming typical rectangles.

a. We first partition either the $x$-axis or the $y$-axis.

☐. Next, using the partition, we form typical rectangles. Then we find the area of each typical rectangle.

b. If we partition the $z$-axis, where $z$ is either $x$ or $y$, the $\Delta z =$ of a typical rectangle.

c. The area of a typical rectangle is.

▶ Disk/Washer Method. Let’s find the volume of the solid of revolution $S$ using the disk/washer method.

a. If the axis of revolution is:
   - the $x$-axis, or parallel to the $x$-axis, then we partition the $x$-axis.
   - the $y$-axis, or parallel to the $y$-axis, then we partition the $y$-axis.

☐. Next, using the partition, we form typical disk/washer’s. Then we find the volume of each typical disk/washer.

b. If we partition the $z$-axis, where $z$ is either $x$ or $y$, the $\Delta z =$ of a typical disk/washer.

c. If we use the disk method, then the volume of a typical disk is:

d. If we use the washer method, then the volume of a typical washer is:

▶ Shell Method. Let’s find the volume of this solid of revolution $S$ using the shell method.

a. If the axis of revolution is:
   - the $x$-axis, or parallel to the $x$-axis, then we partition the $x$-axis.
   - the $y$-axis, or parallel to the $y$-axis, then we partition the $y$-axis.

☐. Next, using the partition, we form typical shells. Then we find the volume of each typical shell.

b. If we partition the $z$-axis, where $z$ is either $x$ or $y$, the $\Delta z =$ of a typical shell.

c. The volume of a typical shell is: