Even/Odd-ness of Maclaurin Polynominals

---

set-up

Throughout this handout, there is the set-up (i.e. notation).

The interval I is centered at 0 and of radius $R > 0$, so $I = (-R, R)$. Let

- $h\colon I \to \mathbb{R}$ be *any* function (naturally, from I into $\mathbb{R}$)
- $g\colon I \to \mathbb{R}$ be *any* differentiable function
- $f\colon I \to \mathbb{R}$ be *any* infinitely differentiable function (i.e. $f^{(n)}(x)$ exists for each $x \in I$ and $n \in \mathbb{N}$)
- $P_N\colon I \to \mathbb{R}$ be N^{th}-order Maclaurin Polyominal of f for each $N \in \mathbb{N}_0$[1], so

$$P_N(x) \;=\; \sum_{n=0}^{N} c_n x^n \qquad \text{where} \qquad c_n \;=\; \frac{f^{(n)}(0)}{n!}\;. \tag{1}$$

Definition 1. even/odd function (what do their graphs look like?)

(1) h is an even function $\iff$ $h(-x) = h(x)$ for each $x \in I$.
(2) h is an odd function $\iff$ $h(-x) = -h(x)$ for each $x \in I$.

Fact 2. These are easy to show.

(1) Examples of even functions: $y = \cos x$, $y = 17$, $y = x^2$, $y = x^4$, $y = x^{\text{any even integer}}$, a constant times an even function, the sum of even functions.
(2) Examples of odd functions: $y = \sin x$, $y = x^1$, $y = x^3$, $y = x^{\text{any odd integer}}$, a constant times an odd function, the sum of odd functions.
(3) A polynomial $y = \sum_{n=0}^{N} c_n x^n$ is
 (a) an even function $\iff$ $c_n = 0$ for the odd $n \in \mathbb{N}_0$ $\iff$[2] *it contains only even powers.*
 (b) an odd function $\iff$ $c_n = 0$ for the even $n \in \mathbb{N}_0$ $\iff$[3] *it contains only odd powers.*
(4) A function can be neither even nor odd, e.g. $y = x - 1$.
(5) If h is an odd function, then $h(0) = 0$.[4]

Fact 3. even/odd and derivatives (think of the graphs and examples)

(1) If g is an even function, then g' is an odd function.
(2) If g is an odd function, then g' is an even function.

Proof. Let $g\colon I \to \mathbb{R}$ be even. Then for each $x \in I$,

$$g'(x) \;=\; \lim_{t\to 0} \frac{g(x+t) - g(x)}{t} \;\overset{g\text{ is even}}{=}\; \lim_{t\to 0} \frac{g(-x-t) - g(-x)}{t}$$

and letting $\tilde{t} = -t$, and so $t = -\tilde{t}$ and noting that $-\tilde{t} \to 0$ if and only if $\tilde{t} \to 0$,

$$= \lim_{-\tilde{t}\to 0} \frac{g\left(-x+\tilde{t}\right) - g(-x)}{-\tilde{t}} \;=\; -\lim_{\tilde{t}\to 0} \frac{g\left(-x+\tilde{t}\right) - g(-x)}{\tilde{t}} \;=\; -g'(-x)\;.$$

So g' is odd. So, if g is even, then g' is odd. So part (1) holds.

Similarly, part (2) holds (i.e., if g is odd, then g' is even). ☐

Now let's apply these simple facts to Maclaurin polynomials.

---

[1]Recal, $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, \ldots\}$.
[2]just loosely speaking
[3]just loosely speaking
[4]Since if h is an odd function, then $h(0) = -h(-0)$ and so $h(0) = -h(0)$.

Fact 4. even/odd and Maclaurin Polynomials

(1) Let f be an even function. Then, for each odd integer $n \in \mathbb{N}_0$,
   (a) $f^{(n)}$ is an odd function
   (b) $f^{(n)}(0) \;=\; 0$
   (c) $c_n \;=\; 0$.
   So each of f's Maclaurin polynonials $\{y = P_N(x)\}_{n=0}^{\infty}$ *contains only even powers.*

(2) Let f be an odd function. Then, for each even integer $n \in \mathbb{N}_0$,
   (a) $f^{(n)}$ is an odd function
   (b) $f^{(n)}(0) \;=\; 0$
   (c) $c_n \;=\; 0$.
   So each of f's Maclaurin polynonials $\{y = P_N(x)\}_{n=0}^{\infty}$ *contains only odd powers.*

Proof. Consider an even f. Then by Fact **??**, we have the following implications.

$$f \text{ is even} \implies f^{(1)} \text{ is odd} \implies f^{(2)} \text{ is even} \implies f^{(3)} \text{ is odd} \implies f^{(4)} \text{ is even} \implies \cdots .$$

So part (1a) holds.

Now consider an odd f. We get the following implications.

$$f^{(0)} \text{ is odd} \implies f^{(1)} \text{ is even} \implies f^{(2)} \text{ is odd} \implies f^{(3)} \text{ is even} \implies f^{(4)} \text{ is odd} \implies \cdots .$$

So part (2a) holds.

In both parts (1) and (2):

- parts (b) follows from Fact **??** part (5)
- parts (c) follows from the definition of c_n, which is in equation (**??**)
- the closing comment about the Maclaurin polynomials P_N's follow from Fact **??** part (3). □

Now let's see what can be said if the Maclaurin polynomials $\{P_{\mathbb{N}}\}_{N=0}^{\infty}$ converge to f.

Fact 5. even/odd and Maclaurin Series
Consider now the case that the Maclaurin polynomials $\{P_N\}_{N=0}^{\infty}$ converge to f, i.e., for each point $x \in I$,

$$\lim_{N\to\infty} P_N(x) \;=\; f(x) \; . \tag{2}$$

(Note the above limit is a limit of the sequence $\{P_N(x)\}_{N=0}^{\infty}$ of real numbers.) Then we say that the Maclaurin series

$$P_\infty(x) \;=\; \sum_{n=0}^{\infty} c_n x^n \qquad \text{, where } \; c_n \;=\; \frac{f^{(n)}(0)}{n!} \; ,$$

converges to f.

(1) If $y = P_\infty(x)$ *contains only even powers*, then f is an even function.
(2) If $y = P_\infty(x)$ *contains only odd powers*, then f is an odd function.

Proof. Let the Maclaurin series P_∞ converge to f. Assume that P_∞ *contains only even powers.* Then each Maclaurin polynomial P_N contains only even powers, and so, by Fact **??** part(3), is an even function. So

$$f(x) \overset{\text{by (??)}}{=} \lim_{N\to\infty} P_N(x) \overset{P_N \text{ is even}}{=} \lim_{N\to\infty} P_N(-x) \overset{\text{by (??)}}{=} f(-x) \; .$$

So f is an even function. So part (1) holds. Part (2) is shown similarly. □

Now, all this should help you remember the *Commonly Used Series.*