Review of some needed Trig. Identities for Integration

- Your answers should be an angle in **RADIANS**.
  - arccos($\frac{1}{2}$) = $\frac{\pi}{3}$
  - arccos($-\frac{1}{2}$) = $\frac{2\pi}{3}$
  - arcsin($\frac{1}{2}$) = $\frac{\pi}{6}$
  - arcsin($-\frac{1}{2}$) = $\frac{-\pi}{6}$
  - Can you do similar problems?

- Double-angle formulas. Your answer should involve trig functions of θ, and not of 2θ.
  - $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
  - $\sin(2\theta) = 2\sin\theta\cos\theta$.

- Half-angle formulas. Your answer should involve $\cos(2\theta)$.
  - $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
  - $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

- Since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween:
  - tangent (i.e., tan) and secant (i.e., sec) is $1+\tan^2\theta = \sec^2\theta$.
  - cotangent (i.e., cot) and cosecant (i.e., csc) is $1+\cot^2\theta = \csc^2\theta$.

Remember Your Calculus I Integration Basics?

- $\int \frac{du}{u} \overset{u\neq 0}{=} \ln|u| + C$
- $\int u^n\,du \overset{n\neq -1}{=} \frac{u^{n+1}}{n+1} + C$
- $\int e^u\,du = e^u + C$
- $\int a^u\,du \overset{a\neq 1}{=} \frac{a^u}{\ln a} + C$
- $\int \cos u\,du = \sin u + C$
- $\int \sec^2 u\,du = \tan u + C$
- $\int \sec u\tan u\,du = \sec u + C$
- $\int \sin u\,du = -\cos u + C$
- $\int \csc^2 u\,du = -\cot u + C$
- $\int \csc u\cot u\,du = -\csc u + C$
- $\int \tan u\,du = \ln|\sec u| \overset{or}{=} -\ln|\cos u| + C$
- $\int \cot u\,du = -\ln|\csc u| \overset{or}{=} \ln|\sin u| + C$
- $\int \sec u\,du = \ln|\sec u + \tan u| \overset{or}{=} -\ln|\sec u - \tan u| + C$
- $\int \csc u\,du = -\ln|\csc u + \cot u| \overset{or}{=} \ln|\csc u - \cot u| + C$
- $\int \frac{1}{\sqrt{a^2-u^2}}\,du \overset{a>0}{=} \sin^{-1}\left(\frac{u}{a}\right) + C$
- $\int \frac{1}{a^2+u^2}\,du \overset{a>0}{=} \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
- $\int \frac{1}{u\sqrt{u^2-a^2}}\,du \overset{a>0}{=} \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

Integration from Calculus II

- **.** Integration by parts formula: $\int u\,dv = $ $uv - \int v du$
- **.** To integrate $\frac{f(x)}{g(x)}$, where f and g are polyonomials, 1st find its Partial Fraction Decomposition (PDF).
  - If [degree of f] $\geq$ [degree of g], then one must first does long division .
  - If [degree of f] $<$ [degree of g] ⟨i.e., have strictly bigger bottoms⟩ then first factor $y = g(x)$ into:
    - $*$ linear factors $px + q$ and
    - $*$ irreducible quadratic factors $ax^2 + bx + c$ (to be sure it's irreducible, you need $b^2 - 4ac < 0$) .

Next, collect up like terms and follow the following rules.

Rule 1: For each factor of the form $(px+q)^m$ where $m \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of m partial fractions of the form, where each A_i is a real number,

$$\frac{A_1}{(px+q)^1} + \frac{A_2}{(px+q)^2} + \ldots + \frac{A_m}{(px+q)^m} .$$

Rule 2: For each factor of the form $(ax^2 + bx + c)^n$ where $n \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of n partial fractions of the form, where the A_i's and B_i's are real number,

$$\frac{A_1x + B_1}{(ax^2 + bx + c)^1} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + d\frac{A_nx+B_n}{(ax^2+bx+c)^n} .$$

- **.** Trig. Substitution. (Recall that the *integrand* is the function you are integrating.) Here, a is a constant and $a > 0$.
  - if the integrand involves $a^2 - u^2$, then one makes the substitution $u = $ $a \sin\theta$.
  - if the integrand involves $a^2 + u^2$, then one makes the substitution $u = $ $a \tan\theta$.
  - if the integrand involves $u^2 - a^2$, then one makes the substitution $u = $ $a \sec\theta$.

Improper Integrals

0. Fill-in-the boxes. Below, $a, b, c \in \mathbb{R}$ with $a < c < b$.

- **.** If $f\colon [0, \infty) \to \mathbb{R}$ is continuous, then we define the improper integral $\int_0^\infty f(x)\,dx$ by

$$\int_0^\infty f(x)\,dx = \lim_{t\to\infty} \int_0^t f(x)\,dx .$$

- **.** If $f\colon (-\infty, 0] \to \mathbb{R}$ is continuous, then we define the improper integral $\int_{-\infty}^0 f(x)\,dx$ by

$$\int_{-\infty}^0 f(x)\,dx = \lim_{t\to-\infty} \int_t^0 f(x)\,dx .$$

- If $f\colon (-\infty,\infty) \to \mathbb{R}$ is continuous, then we define the improper integral $\int_{-\infty}^{\infty} f(x)\, dx$ by

$$\int_{-\infty}^{\infty} f(x)\, dx = \boxed{\left[\lim_{t\to-\infty} \int_t^0 f(x)\, dx\right] + \left[\lim_{s\to\infty} \int_0^s f(x)\, dx\right]}\,.$$

- If $f\colon (a,b] \to \mathbb{R}$ is continuous, then we define the improper integral $\int_a^b f(x)\, dx$ by

$$\int_a^b f(x)\, dx = \boxed{\lim_{t\to a^+} \int_t^b f(x)\, dx}\,.$$

- If $f\colon [a,b) \to \mathbb{R}$ is continuous, then we define the improper integral $\int_a^b f(x)\, dx$ by

$$\int_a^b f(x)\, dx = \boxed{\lim_{t\to b^-} \int_a^t f(x)\, dx}\,.$$

- If $f\colon [a,c) \cup (c,b] \to \mathbb{R}$ is continuous, then we define the improper integral $\int_a^b f(x)\, dx$ by

$$\int_a^b f(x)\, dx = \boxed{\left[\lim_{t\to c^-} \int_a^t f(x)\, dx\right] + \left[\lim_{s\to c^+} \int_s^b f(x)\, dx\right]}\,.$$

- An improper integral as above *converges* precisely when

each of the limits involves converges to a **finite** number.

- An improper integral as above *diverges* precisely when

the improper integral does not converge.

- An improper integral as above *diverges to* ∞ precisely when

at least one of the involved limits diverges to ∞ AND
each of the other involved limits either diverges to ∞ or converges to a **finite** number.

- An improper integral as above *diverges to* $^-\infty$ precisely when

at least one of the involved limits diverges to $^-\infty$ AND
each of the other involved limits either diverges to $^-\infty$ or converges to a **finite** number.

Sequences

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Complete the below sentences.
  - The limit of $\{a_n\}_{n=1}^{\infty}$ is the real number L provided for each $\epsilon > 0$ there exists a natural number N so that if the natural number n satisfies $\underline{\ n\ } > \underline{\ N\ }$ then $\underline{|L - a_n|} < \underline{\ \epsilon\ }$.
  - If the limit of $\{a_n\}_{n=1}^{\infty}$ is $L \in \mathbb{R}$, then we denote this by $\underline{\lim_{n\to\infty} a_n = L}$.
  - $\{a_n\}_{n=1}^{\infty}$ converges provided $\underline{\text{there exists a real number } L \text{ so that } \lim_{n\to\infty} a_n = L}$.
  - $\{a_n\}_{n=1}^{\infty}$ diverges provided $\{a_n\}_{n=1}^{\infty}$ $\underline{\text{does not converge}}$.

- Practice taking basic limits. (Important, e.g., for Ratio and Root Tests.)
  - $\displaystyle\lim_{n\to\infty} \frac{5n^{17} + 6n^2 + 1}{7n^{18} + 9n^3 + 5} = \underline{\ 0\ }$
  - $\displaystyle\lim_{n\to\infty} \frac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5} = \underline{\tfrac{36}{4} \text{ or } 9}$
  - $\displaystyle\lim_{n\to\infty} \frac{-5n^{18} + 6n^2 + 1}{7n^{17} + 9n^3 + 5} = \underline{\text{DNE or } -\infty}$
  - $\displaystyle\lim_{n\to\infty} \sqrt{\frac{36n^{17} - 6n^2 - 1}{4n^{17} + 9n^3 + 5}} = \underline{\sqrt{\tfrac{36}{4}} \text{ or } 3}$
  - Can you do similar problems?

- Commonly Occurring Limits ⟨Thomas Book §10.1, Theorem 5 page 578⟩

(1) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = \boxed{0}$

(2) $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = \boxed{1}$

(3) $\displaystyle\lim_{n\to\infty} c^{1/n} = \boxed{1}$ $(c > 0)$

(4) $\displaystyle\lim_{n\to\infty} c^n = \boxed{0}$ $(|c| < 1)$

(5) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = \boxed{e^c}$ $(c \in \mathbb{R})$

(6) $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = \boxed{0}$ $(c \in \mathbb{R})$

- Let $-\infty < r < \infty$. (Needed for Geometric Series. Warning, don't confuse sequences with series.)
  - If $|r| < 1$, then $\displaystyle\lim_{n\to\infty} r^n = \boxed{0}$.
  - If $r = 1$, then $\displaystyle\lim_{n\to\infty} r^n = \boxed{1}$.
  - If $r > 1$, then $\displaystyle\lim_{n\to\infty} r^n = \boxed{\text{DNE (tends to } \infty)}$.
  - If $r = -1$, then $\displaystyle\lim_{n\to\infty} r^n = \boxed{\text{DNE (oscillates between 1 and } -1)}$.
  - If $r < -1$, then $\displaystyle\lim_{n\to\infty} r^n = \boxed{\text{DNE } (r^{2n} \to \infty \text{ while } r^{2n+1} \to -\infty)}$.

Series

▶. In this section, all series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

- For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums, so $s_N = \sum_{n=1}^{N} a_n$. Then the formal series $\sum a_n$:
  - converges if and only if the $\lim_{N\to\infty} s_N$ exists in $\mathbb{R}$
  - converges to $L \in \mathbb{R}$ if and only if the $\lim_{N\to\infty} s_N$ exists in $\mathbb{R}$ and equals $L \in \mathbb{R}$
  - diverges if and only if the $\lim_{N\to\infty} s_N$ does not exist in $\mathbb{R}$.

  Now assume, furthermore, that $a_n \geq 0$ for each n. Then the sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums either
  - is bounded above (by some finite number), in which case the series $\sum a_n$ converges

  or
  - is not bounded above (by some finite number), in which case the series $\sum a_n$ diverges to $^{+}\infty$.

- State the n^{th}**-term test** for an arbitrary series $\sum a_n$.

  If $\lim_{n\to\infty} a_n \neq 0$ (which includes the case that $\lim_{n\to\infty} a_n$ does not exist), then $\sum a_n$ diverges .

- Fix $r \in \mathbb{R}$. For $N \geq 17$, let $s_N = \sum_{\mathbf{n=17}}^{N} r^n$ (Note the sum starts at 17). Then, for $N > 17$,
  - $s_N =$ $r^{17} + r^{18} + \ldots + r^N$ (your answer can have …'s but not a $\sum$ sign)
  - $r\, s_N =$ $r^{18} + \ldots + r^N + r^{N+1}$ (your answer can have …'s but not a $\sum$ sign)
  - $(1-r)\, s_N =$ $r^{17} - r^{N+1}$ (your answer should have neither …'s nor a $\sum$ sign)
  - and if $r \neq 1$, then $s_N =$ $\frac{r^{17} - r^{N+1}}{1-r}$ (your answer should have neither …'s nor a $\sum$ sign)

- **Geometric Series** where $-\infty < r < \infty$. The series $\sum r^n$ (hint: look at the previous questions):
  - converges if and only if $|r| < 1$
  - diverges if and only if $|r| \geq 1$.

- p**-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$
  - converges if and only if $p > 1$.
  - diverges if and only if $p \leq 1$.

  This can be shown by using the integral test ⟨here, name the test one uses⟩ and comparing (the hard to compute series) $\sum \frac{1}{n^p}$ to (the easy to compute improper integral) $\int_{x=1}^{\infty} \frac{1}{x^p}\, dx$.

Tests for Positive-Termed Series
(so for $\sum a_n$ where $a_n \geq 0$)

0.1. State the **Integral Test with Remainder Estimate** for a positive-termed series $\sum a_n$.

Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

(1) $a_n = f(n)$ for each $n \in \mathbb{N}$

(2) f is a [positive] function

(3) f is a [continuous] function

(4) f is a [decreasing (nonincreasing is also ok)] function.

Then

- $\sum a_n$ converges if and only if [$\int_{x=1}^{x=\infty} f(x)\,dx$] converges.
- and if $\sum a_n$ converges, then

$$0 \quad \leq \quad \left(\sum_{k=1}^{\infty} a_k\right) - \left(\sum_{k=1}^{N} a_k\right) \quad \leq \quad \boxed{\int_{x=N}^{x=\infty} f(x)\,dx} \; .$$

0.2. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If [$0 \leq a_n \leq c_n$ (only $a_n \leq c_n$ is also ok b/c given $a_n \geq 0$)] when $n \geq 17$ and [$\sum c_n$ converges], then $\sum a_n$ converges.
- If [$0 \leq d_n \leq a_n$ (need $0 \leq d_n$ part here)] when $n \geq 17$ and [$\sum d_n$ diverges], then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.3. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then [$[\, \sum b_n$ converges $\iff$ $\sum a_n$ converges $]$]
- If $L = 0$, then [$[\, \sum b_n$ converges $\implies$ $\sum a_n$ converges $]$] .
- If $L = \infty$, then [$[\, \sum b_n$ diverges $\implies$ $\sum a_n$ diverges $]$] .

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{\frac{a_n}{b_n}\right\}_n$ converges.

0.4. **Helpful Intuition** Fill in the 3 boxes using: e^x, $\ln x$, x^q. Use each once, and only once.

Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \geq N_q$ then

$$\boxed{\ln x} \quad \leq \quad \boxed{x^q} \quad \leq \quad \boxed{e^x} \; .$$

Tests for Arbitrary-Termed Series
(so for $\sum a_n$ where $-\infty < a_n < \infty$)

0.5. By definition, for an arbitrary series $\sum a_n$, (fill in these 3 boxes with convergent or divergent).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ is [convergent].
- $\sum a_n$ is conditionally convergent if and only if

  $\sum a_n$ is [convergent] and $\sum |a_n|$ is [divergent].
- $\sum a_n$ is divergent if and only if $\sum a_n$ is divergent.

0.6. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

- If [$\rho < 1$] then $\sum a_n$ converges absolutely.
- If [$\rho > 1$] then $\sum a_n$ diverges.
- If [$\rho = 1$] then the test is inconclusive.

0.7. State the **Alternating Series Test** (AST) & **Alternating Series Estimation Theorem**.

Let

(1) $u_n \geq 0$ for each $n \in \mathbb{N}$

(2) $\lim_{n\to\infty} u_n =$ [0]

(3) u_n [$>$ (also ok $\geq$)] u_{n+1} for each $n \in \mathbb{N}$.

Then

- [the series $\sum (-1)^n u_n$ converges. (also ok: $\sum (-1)^{n+1} u_n$ converges or $\sum (-1)^{n-1} u_n$ converges)]
- and we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty} (-1)^n u_n$ by the finite sum $\sum_{k=1}^{N} (-1)^k u_k$ and the error (i.e. remainder) satisfies

$$\left| \sum_{k=1}^{\infty} (-1)^k u_k \; - \; \sum_{k=1}^{N} (-1)^k u_k \right| \leq \boxed{u_{N+1}} .$$

Power Series

Condsider a (formal) power series

$$h(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n , \tag{1.1}$$

with radius of convergence $R \in [0, \infty]$.

(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

- The choices for the next 4 boxes are: AC, CC, DIVG, anything. Here,

  AC stands for: *always absolutely convergent*

  CC stands for: *always conditionally convergent*

  DIVG stands for: *is always divergent*

  anything stands for: *can do anything, i.e., there are examples showing that it can be AC, CC, or DIVG.*

  (1) At the center $x = x_0$, the power series in (1.1) [AC].

  (2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series in (1.1) [AC].

  (3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$, the power series in (1.1) [DIVG].

  (4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series in (1.1) [anything].

- For this part, fill in the 7 boxes.

  Let $R > 0$ and consider the function $y = h(x)$ defined by the power series in (1.1).

  (1) The function $y = h(x)$ is always differentiable on the interval [$(x_0 - R, x_0 + R)$]

  (make this interval as large as it can be, but still keeping the statement true).

  Furthermore, if x is in this interval, then

$$h'(x) = \sum_{n=\boxed{1}}^{\infty} \boxed{n\, a_n (x - x_0)^{n-1}} . \tag{1.2}$$

  What can you say about the radius of convergence of the power series in (1.2)?

  [The power series in (1.2) has the same raduis of convergence as the power series in (1.1).]

  (2) The function $y = h(x)$ always has an antiderivative on the interval [$(x_0 - R, x_0 + R)$]

  (make this interval as large as it can be, but still keeping the statement true).

  Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\, dx = \sum_{n=\boxed{0}}^{\infty} \boxed{\frac{a_n}{n+1} (x - x_0)^{n+1}} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} .$$

Taylor/Maclaurin Polynomials and Series

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. The formula for c_n is

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

b. In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N$$

c. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

d. In open form (i.e., with ... and without a $\sum$-sign)

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$$

e. In closed form (i.e., with a $\sum$-sign and without ...)

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x - x_0)^{(N+1)}$$ for some c between x and x_0.

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = 0$.

Commonly Used Taylor Series

▶. Here, *expansion* refers to the power series expansion that is the Maclaurin series.

- An expansion for $y = e^x$ is $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

- An expansion for $y = \cos x$ is $\boxed{\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

- An expansion for $y = \sin x$ is $\boxed{\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$, which is valid precisely when $x \in \boxed{(-\infty, \infty)}$.

- An expansion for $y = \frac{1}{1-x}$ is $\boxed{\displaystyle\sum_{n=0}^{\infty} x^n}$, which is valid precisely when $x \in \boxed{(-1, 1)}$.

- An expansion for $y = \ln(1+x)$ is $\boxed{\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$, which is valid precisely when $x \in \boxed{(-1, 1]}$.

- An expansion for $y = \arctan x$ is $\boxed{\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$, which is valid precisely when $x \in \boxed{[-1, 1]}$.

Parametric Curves

In this part, fill in the 4 boxes. Consider the curve $\mathcal{C}$ parameterized by

$$x = x(t)$$
$$y = y(t)$$

for $a \le t \le b$.

1) Express $\frac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\frac{dy}{dx} = \boxed{\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}}$

2) The tangent line to $\mathcal{C}$ when $t = t_0$ is $y = mx + b$ where m is $\boxed{\frac{dy}{dx}}$ evaluated at $t = t_0$.

3) Express $\frac{d^2y}{dx^2}$ using derivatives with respect to t. Answer: $\frac{d^2y}{dx^2} = \boxed{\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$

4) The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

$$\text{Arc Length} = \boxed{\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt}$$

Polar Coordinates

▶. Here, CC stands for *Cartresian coordinates* while PC stands for *polar coordinates*.

- A point with PC (r, θ) also has PC $\left(\boxed{r}, \theta + 2\pi\right)$ as well as $\left(\boxed{-r}, \theta + \pi\right)$.
- A point $P \in \mathbb{R}^2$ with CC (x, y) and PC (r, θ) satisfies the following.

$$x = \boxed{r\,\cos\theta} \quad \& \quad y = \boxed{r\,\sin\theta} \quad \& \quad r^2 = \boxed{x^2 + y^2} \quad \& \quad \boxed{\tan\theta} = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases}.$$

- The period of $f(\theta) = \cos(k\theta)$ and of $f(\theta) = \sin(k\theta)$ is $\boxed{\frac{2\pi}{k}}$.

  To sketch these graphs, we divide the period by $\boxed{4}$ and make *the chart*,

  in order to detect the $\boxed{\text{max/min/zero's of the function } r = f(\theta)}$.

- Now consider a sufficiently *nice* function $r = f(\theta)$ which determines a curve in the plane. The the area bounded by polar curves $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \boxed{\tfrac{1}{2}\,[f(\theta)]^2}\; d\theta\,.$$

  The arc length of the polar curves $r = f(\theta)$ is

$$\text{Arc Length} = \int_{\theta=\alpha}^{\theta=\beta} \boxed{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}\; d\theta\,.$$