Review of some needed Trig. Identities for Integration

- Your answers should be an angle in **RADIANS**.
  - arccos($\frac{1}{2}$) = ________
  - arccos(-$\frac{1}{2}$) = ________
  - arcsin($\frac{1}{2}$) = ________
  - arcsin(-$\frac{1}{2}$) = ________
  - Can you do similar problems?
- Double-angle formulas. Your answer should involve trig functions of θ, and not of 2θ.
  - $\cos(2\theta) =$ [ ]
  - $\sin(2\theta) =$ [ ] .
- Half-angle formulas. Your answer should involve $\cos(2\theta)$.
  - $\cos^2(\theta) =$ [ ]
  - $\sin^2(\theta) =$ [ ]
- Since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween:
  - tangent (i.e., tan) and secant (i.e., sec) is ________________ .
  - cotangent (i.e., cot) and cosecant (i.e., csc) is ________________ .

Remember Your Calculus I Integration Basics?

- $\int \frac{du}{u} \overset{u\neq 0}{=}$ ________________ + C
- $\int u^n \, du \overset{n\neq -1}{=}$ ________________ +C
- $\int e^u \, du =$ ________________ +C
- $\int a^u \, du \overset{a\neq 1}{=} =$ ________________ + C
- $\int \cos u \, du =$ ________________ $+ C$
- $\int \sec^2 u \, du =$ ________________ $+ C$
- $\int \sec u \tan u \, du =$ ________________ $+ C$
- $\int \sin u \, du =$ ________________ $+ C$
- $\int \csc^2 u \, du =$ ________________ $+ C$
- $\int \csc u \cot u \, du =$ ________________ $+ C$
- $\int \tan u \, du =$ ________________ $+ C$
- $\int \cot u \, du =$ ________________ $+ C$
- $\int \sec u \, du =$ ________________ $+ C$
- $\int \csc u \, du =$ ________________ $+ C$
- $\int \frac{1}{\sqrt{a^2-u^2}} \, du \overset{a\geq 0}{=}$ ________________ $+ C$
- $\int \frac{1}{a^2+u^2} \, du \overset{a\geq 0}{=}$ ________________ $+ C$
- $\int \frac{1}{u\sqrt{u^2-a^2}} \, du \overset{a\geq 0}{=}$ ________________ $+ C$

Integration from Calculus II

- Integration by parts formula: $\int u\, dv \;=$ ____

- To integrate $\frac{f(x)}{g(x)}$, where f and g are polyonomials, 1st find its ________________________________ (PDF).
  - If [degree of f] $\geq$ [degree of g], then one must first does ____________________________ .
  - If [degree of f] $<$ [degree of g] ⟨i.e., have strictly bigger bottoms⟩ then first factor $y = g(x)$ into:
    - * [   ] factors $px + q$ and
    - * irreducible [   ] factors $ax^2 + bx + c$ (to be sure it's irreducible, you need [   ]) .

  Next, collect up like terms and follow the following rules.

  Rule 1: For each factor of the form $(px+q)^m$ where $m \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of [   ] partial fractions of the form, where each A_i is a real number,

  [   ] .

  Rule 2: For each factor of the form $(ax^2 + bx + c)^n$ where $n \geq 1$, the decomposition of $y = \frac{f(x)}{g(x)}$ contains a sum of [   ] partial fractions of the form, where the A_i's and B_i's are real number,

  [   ] .

- Trig. Substitution. (Recall that the *integrand* is the function you are integrating.) Here, a is a constant and $a > 0$.
  - if the integrand involves $a^2 - u^2$, then one makes the substitution $u \;=$ ____________________ .
  - if the integrand involves $a^2 + u^2$, then one makes the substitution $u \;=$ ____________________ .
  - if the integrand involves $u^2 - a^2$, then one makes the substitution $u \;=$ ____________________ .

Improper Integrals

0. Fill-in-the boxes. Below, $a, b, c \in \mathbb{R}$ with $a < c < b$.

- If $f\colon [0, \infty) \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_0^{\infty} f(x)\, dx$ by

$$\int_0^{\infty} f(x)\, dx = \boxed{}\,.$$

- If $f\colon (-\infty, 0] \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_{-\infty}^{0} f(x)\, dx$ by

$$\int_{-\infty}^{0} f(x)\, dx = \boxed{}\,.$$

- If $f\colon (-\infty, \infty) \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx$ by

$$\int_{-\infty}^{\infty} f(x)\, dx = \boxed{} \,.$$

- If $f\colon (a, b] \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_{a}^{b} f(x)\, dx$ by

$$\int_{a}^{b} f(x)\, dx = \boxed{} \,.$$

- If $f\colon [a, b) \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_{a}^{b} f(x)\, dx$ by

$$\int_{a}^{b} f(x)\, dx = \boxed{} \,.$$

- If $f\colon [a, c) \cup (c, b] \to \mathbb{R}$ is continuous, then we define the improper integral $\displaystyle\int_{a}^{b} f(x)\, dx$ by

$$\int_{a}^{b} f(x)\, dx = \boxed{} \,.$$

- An improper integral as above *converges* precisely when

- An improper integral as above *diverges* precisely when

- An improper integral as above *diverges to* ∞ precisely when

- An improper integral as above *diverges to* ${}^{-}\infty$ precisely when

Sequences

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Complete the below sentences.
  - The limit of $\{a_n\}_{n=1}^{\infty}$ is the real number L provided for each $\epsilon > 0$ there exists a natural number N so that if the natural number n satisfies ____ $>$ ____ then _________ $<$ ____ .
  - If the limit of $\{a_n\}_{n=1}^{\infty}$ is $L \in \mathbb{R}$, then we denote this by ____________________________ .
  - $\{a_n\}_{n=1}^{\infty}$ converges provided ____ .
  - $\{a_n\}_{n=1}^{\infty}$ diverges provided $\{a_n\}_{n=1}^{\infty}$ ____ .

- Practice taking basic limits. (Important, e.g., for Ratio and Root Tests.)
  - $\lim\limits_{n\to\infty} \dfrac{5n^{17}+6n^2+1}{7n^{18}+9n^3+5} =$ ____________
  - $\lim\limits_{n\to\infty} \dfrac{36n^{17}-6n^2-1}{4n^{17}+9n^3+5} =$ ____________
  - $\lim\limits_{n\to\infty} \dfrac{-5n^{18}+6n^2+1}{7n^{17}+9n^3+5} =$ ____________
  - $\lim\limits_{n\to\infty} \sqrt{\dfrac{36n^{17}-6n^2-1}{4n^{17}+9n^3+5}} =$ ____________
  - Can you do similar problems?

- Commonly Occurring Limits ⟨Thomas Book §10.1, Theorem 5 page 578⟩

  (1) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} =$ [ ]

  (2) $\lim\limits_{n\to\infty} \sqrt[n]{n} =$ [ ]

  (3) $\lim\limits_{n\to\infty} c^{1/n} =$ [ ] $(c > 0)$

  (4) $\lim\limits_{n\to\infty} c^n =$ [ ] $(|c| < 1)$

  (5) $\lim\limits_{n\to\infty} \left(1+\frac{c}{n}\right)^n =$ [ ] $(c \in \mathbb{R})$

  (6) $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} =$ [ ] $(c \in \mathbb{R})$

- Let $-\infty < r < \infty$. (Needed for Geometric Series. Warning, don't confuse sequences with series.)
  - If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r = 1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r > 1$, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r =$ -1, then $\lim\limits_{n\to\infty} r^n =$ [ ] .
  - If $r <$ -1, then $\lim\limits_{n\to\infty} r^n =$ [ ] .

Series

▶. In this section, all series $\sum$ are understood to be $\sum_{n=1}^{\infty}$, unless otherwise indicated.

- For a formal series $\sum_{n=1}^{\infty} a_n$, where each $a_n \in \mathbb{R}$, consider the corresponding sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums, so $s_N = \sum_{n=1}^{N} a_n$. Then the formal series $\sum a_n$:
  - converges if and only if ______________________________
  - converges to $L \in \mathbb{R}$ if and only if ______________________________
  - diverges if and only if ______________________________ .

  Now assume, furthermore, that $a_n \geq 0$ for each n. Then the sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums either
  - is bounded above (by some finite number), in which case the series $\sum a_n$ ______________

  or
  - is not bounded above (by some finite number), in which case the series $\sum a_n$ ______________ .

- State the $n^{\mathbf{th}}$**-term test** for an arbitrary series $\sum a_n$.

- Fix $r \in \mathbb{R}$. For $N \geq 17$, let $s_N = \sum_{\mathbf{n=17}}^{N} r^n$ (Note the sum starts at 17). Then, for $N > 17$,
  - $s_N =$ ______________________________ (your answer can have . . .'s but not a $\sum$ sign)
  - $r\, s_N =$ ______________________________ (your answer can have . . .'s but not a $\sum$ sign)
  - $(1 - r)\, s_N =$ ______________________ (your answer should have neither . . .'s nor a $\sum$ sign)
  - and if $r \neq 1$, then $s_N =$ ______________ (your answer should have neither . . .'s nor a $\sum$ sign)

- **Geometric Series** where $-\infty < r < \infty$. The series $\sum r^n$ (hint: look at the previous questions):
  - converges if and only if [ ]
  - diverges if and only if [ ] .

- p**-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$
  - converges if and only if [ ] .
  - diverges if and only if [ ] .

  This can be shown by using the ______________ ⟨here, name the test one uses⟩ and comparing (the hard to compute series) $\sum \frac{1}{n^p}$ to (the easy to compute improper integral) $\int_{x=1}^{\infty}$ ________ dx .

Tests for Positive-Termed Series
(so for $\sum a_n$ where $a_n \geq 0$)

0.1. State the **Integral Test with Remainder Estimate** for a positive-termed series $\sum a_n$.

Let $f\colon [1, \infty) \to \mathbb{R}$ be so that

(1) $a_n = f(n)$ for each $n \in \mathbb{N}$

(2) f is a ______ function

(3) f is a ______ function

(4) f is a ______ function.

Then

- $\sum a_n$ converges if and only if ______ converges.
- and if $\sum a_n$ converges, then

$$0 \quad \leq \quad \left(\sum_{k=1}^{\infty} a_k \right) - \left(\sum_{k=1}^{N} a_k \right) \quad \leq \quad \boxed{} \,.$$

0.2. State the **Direct Comparison Test** for a positive-termed series $\sum a_n$.

- If ______ when $n \geq 17$ and ______ , then $\sum a_n$ converges.
- If ______ when $n \geq 17$ and ______ , then $\sum a_n$ diverges.

Hint: sing the song to yourself.

0.3. State the **Limit Comparison Test** for a positive-termed series $\sum a_n$.

Let $b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

- If $0 < L < \infty$, then ______
- If $L = 0$, then ______ .
- If $L = \infty$, then ______ .

Goal: cleverly pick positive b_n's so that you know what $\sum b_n$ does (converges or diverges) and the sequence $\left\{ \frac{a_n}{b_n} \right\}_n$ converges.

0.4. Helpful Intuition Fill in the 3 boxes using: e^x, $\ln x$, x^q. Use each once, and only once.

Consider a positive power $q > 0$. There is (some big number) $N_q > 0$ so that if $x \geq N_q$ then

$$\boxed{} \leq \boxed{} \leq \boxed{} \,.$$

Tests for Arbitrary-Termed Series
(so for $\sum a_n$ where $-\infty < a_n < \infty$)

0.5. By definition, for an arbitrary series $\sum a_n$, (fill in these 3 boxes with convergent or divergent).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ is [ ].
- $\sum a_n$ is conditionally convergent if and only if

  $\sum a_n$ is [ ] and $\sum |a_n|$ is [ ].
- $\sum a_n$ is divergent if and only if $\sum a_n$ is divergent.

0.6. State the **Ratio and Root Tests** for arbitrary-termed series $\sum a_n$ with $-\infty < a_n < \infty$. Let

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{or} \qquad \rho = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

- If [ ] then $\sum a_n$ converges absolutely.
- If [ ] then $\sum a_n$ diverges.
- If [ ] then the test is inconclusive.

0.7. State the **Alternating Series Test** (AST) & **Alternating Series Estimation Theorem**.

Let

(1) $u_n \geq 0$ for each $n \in \mathbb{N}$

(2) $\lim_{n\to\infty} u_n =$ [ ]

(3) u_n [ ] u_{n+1} for each $n \in \mathbb{N}$.

Then

- [ ]
- and we can estimate (i.e., approximate) the infinite sum $\sum_{n=1}^{\infty}(-1)^n u_n$ by the finite sum $\sum_{k=1}^{N}(-1)^k u_k$ and the error (i.e. remainder) satisfies

$$\left| \sum_{k=1}^{\infty}(-1)^k u_k - \sum_{k=1}^{N}(-1)^k u_k \right| \leq \boxed{} .$$

Power Series

Condsider a (formal) power series

$$h(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \,, \tag{1.1}$$

with radius of convergence $R \in [0, \infty]$.

(Here $x_0 \in \mathbb{R}$ is fixed and $\{a_n\}_{n=0}^{\infty}$ is a fixed sequence of real numbers.)

Without any other further information on $\{a_n\}_{n=0}^{\infty}$, answer the following questions.

- The choices for the next 4 boxes are: AC, CC, DIVG, anything. Here,

  AC stands for: *always absolutely convergent*

  CC stands for: *always conditionally convergent*

  DIVG stands for: *is always divergent*

  anything stands for: *can do anything, i.e., there are examples showing that it can be AC, CC, or DIVG.*

  (1) At the center $x = x_0$, the power series in (1.1) $\boxed{}$.

  (2) For $x \in \mathbb{R}$ such that $|x - x_0| < R$, the power series in (1.1) $\boxed{}$.

  (3) For $x \in \mathbb{R}$ such that $|x - x_0| > R$, the power series in (1.1) $\boxed{}$.

  (4) If $R > 0$, then for the endpoints $x = x_0 \pm R$, the power series in (1.1) $\boxed{}$.

- For this part, fill in the 7 boxes.

  Let $R > 0$ and consider the function $y = h(x)$ defined by the power series in (1.1).

  (1) The function $y = h(x)$ is always differentiable on the interval $\boxed{}$ (make this interval as large as it can be, but still keeping the statement true).

  Furthermore, if x is in this interval, then

$$h'(x) = \sum_{n=\boxed{}}^{\infty} \boxed{} \,. \tag{1.2}$$

  What can you say about the radius of convergence of the power series in (1.2)?

  $\boxed{}$

  (2) The function $y = h(x)$ always has an antiderivative on the interval $\boxed{}$ (make this interval as large as it can be, but still keeping the statement true).

  Futhermore, if α and β are in this interval, then

$$\int_{x=\alpha}^{x=\beta} h(x)\,dx = \sum_{n=\boxed{}}^{\infty} \boxed{} \Bigg|_{\mathbf{x}=\alpha}^{\mathbf{x}=\beta} \,.$$

Taylor/Maclaurin Polynomials and Series

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. The formula for c_n is

$c_n =$ [ ]

b. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$P_N(x) =$ [ ]

c. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$P_N(x) =$ [ ]

d. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$P_\infty(x) =$ [ ]

e. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$P_\infty(x) =$ [ ]

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$R_N(x) =$ [ ] for some c between [ ] and [ ].

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 =$ [ ].

Commonly Used Taylor Series

▶. Here, *expansion* refers to the power series expansion that is the Maclaurin series.

- An expansion for $y = e^x$ is $\boxed{}$, which is valid precisely when $x \in$ $\boxed{}$.

- An expansion for $y = \cos x$ is $\boxed{}$, which is valid precisely when $x \in$ $\boxed{}$.

- An expansion for $y = \sin x$ is $\boxed{}$, which is valid precisely when $x \in$ $\boxed{}$.

- An expansion for $y = \frac{1}{1-x}$ is $\boxed{}$, which is valid precisely when $x \in$ $\boxed{}$.

- An expansion for $y = \ln(1+x)$ is $\boxed{}$, which is valid precisely when $x \in$ $\boxed{}$.

- An expansion for $y = \arctan x$ is $\boxed{}$, which is valid precisely when $x \in$ $\boxed{}$.

Parametric Curves

In this part, fill in the 4 boxes. Consider the curve $\mathcal{C}$ parameterized by

$$x = x(t)$$
$$y = y(t)$$

for $a \leq t \leq b$.

1) Express $\frac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\frac{dy}{dx} = \boxed{}$

2) The tangent line to $\mathcal{C}$ when $t = t_0$ is $y = mx + b$ where m is $\boxed{}$ evaluated at $t = t_0$.

3) Express $\frac{d^2y}{dx^2}$ using derivatives with respect to t. Answer: $\frac{d^2y}{dx^2} = \boxed{}$

4) The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

Arc Length = $\boxed{}$

Polar Coordinates

▶. Here, CC stands for *Cartresian coordinates* while PC stands for *polar coordinates*.

- A point with PC (r, θ) also has PC $\left(\boxed{}\ , \theta + 2\pi\right)$ as well as $\left(\boxed{}\ , \theta + \pi\right)$.
- A point $P \in \mathbb{R}^2$ with CC (x, y) and PC (r, θ) satisfies the following.

$$x = \boxed{} \quad \& \quad y = \boxed{} \quad \& \quad r^2 = \boxed{} \quad \& \quad \boxed{} = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases}.$$

- The period of $f(\theta) = \cos(k\theta)$ and of $f(\theta) = \sin(k\theta)$ is $\boxed{}$.

  To sketch these graphs, we divide the period by $\boxed{}$ and make *the chart*,

  in order to detect the $\boxed{}$.

- Now consider a sufficiently *nice* function $r = f(\theta)$ which determines a curve in the plane. The the area bounded by polar curves $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \boxed{} \, d\theta \, .$$

The arc length of the polar curves $r = f(\theta)$ is

$$\text{Arc Length} = \int_{\theta=\alpha}^{\theta=\beta} \boxed{} \, d\theta \, .$$