INFINITE SERIES – SUMMARY

GEOMETRIC SERIES: WITH RATIO r (AND $c \neq 0$)

$$\sum_{n=0}^{\infty} c\, r^n = c\left(1 + r + r^2 + r^3 + r^4 + \dots\right) = \begin{cases} \text{converges} & |r| < 1 \\ \text{diverges} & |r| \geqslant 1 \end{cases}$$

Since $\sum_{n=0}^{N} r^n \equiv s_N = \frac{1 - r^{N+1}}{1 - r}$.

p-SERIES

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p = \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \left(\frac{1}{2}\right)^p + \left(\frac{1}{3}\right)^p + \left(\frac{1}{4}\right)^p + \dots = \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leqslant 1 \end{cases}$$

Show this via Integral Test.
If $p = 1$, it's called the harmonic series.

n^{th}-TERM TEST FOR DIVERGENCE

The Test:	If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ DNE, then $\sum a_n$ diverges.
Because:	If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
Warning:	If $\lim_{n\to\infty} a_n = 0$, then it is possible that $\sum a_n$ converges and it is possible that $\sum a_n$ diverges.
Remark:	The n^{th} can show divergence but can NOT show convergence.

DEFINITIONS

$\sum a_n$ is absolutely convergent $\iff$ [$\sum |a_n|$ converges]

$\sum a_n$ is conditionally convergent $\iff$ [$\sum |a_n|$ diverges AND $\sum a_n$ converges]

$\sum a_n$ is divergent $\iff$ [$\sum a_n$ diverges]

BIG THEOREM

Theorem:
If $\sum |a_n|$ converges, then $\sum a_n$ converges.
So we get for free:
If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.

MUTUALLY EXCLUSIVE AND EXHAUSTIVE POSSIBILITIES

$\sum a_n$ is absolutely convergent $\iff$ [$\sum |a_n|$ converges $\overset{\text{implies}}{\Longrightarrow}$ $\sum a_n$ converges]

$\sum a_n$ is conditionally convergent $\iff$ [$\sum |a_n|$ diverges AND $\sum a_n$ converges]

$\sum a_n$ is divergent $\iff$ [$\sum a_n$ diverges $\overset{\text{implies}}{\Longrightarrow}$ $\sum |a_n|$ diverges]

PROBLEM: we need to figure out if an infinite series $\sum a_n$ is: absolutely convergent, conditionally convergent, or divergent.
SOLUTION: we apply one of the below TESTS that will give us the answer. Which one ... well, pattern recognition time. Sometimes more than one test will work! For some of the tests, we need to find the appropriate $\sum b_n$, which is usually a well-known series (like a geometric series or p-series) that we know whether it converges or diverges.

NAME	STATEMENT OF TEST
	POSITIVE-TERMED SERIES TESTS $\sum a_n$ where $a_n \geqslant 0 \ \forall n \in \mathbb{N}$
Key Idea	$\sum a_n$ converge $\iff \{s_N\}_{N=1}^{\infty}$ is bounded above (since $a_n \geqslant 0 \iff s_n \nearrow$)
Integral Test	Let $f\colon [1,\infty) \to \mathbb{R}$ be continuous, positive, and nonincreasing function with $f(n) = a_n \forall n \in \mathbb{N}$. Then $\left[\sum a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}\right]$.
Comparison Test (CT)	$[\,0 \leqslant a_n \leqslant b_n\ \forall n \geqslant N_0\ \&\ \sum b_n \text{ conv.}\,] \Longrightarrow [\,\sum a_n \text{ conv.}\,]$ $[\,0 \leqslant b_n \leqslant a_n\ \forall n \geqslant N_0\ \&\ \sum b_n \text{ divg.}\,] \Longrightarrow [\,\sum a_n \text{ divg.}\,]$
Limit Comparison Test (LCT)	Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then $[\,\sum a_n \text{ conv.} \iff \sum b_n \text{ conv.}\,]$ (you DO need to memorize this one) If $L = 0$, then $[\,\sum b_n \text{ conv.} \Longrightarrow \sum a_n \text{ conv.}\,]$ (you do not have to memorize this one) If $L = \infty$, then $[\,\sum b_n \text{ divg.} \Longrightarrow \sum a_n \text{ divg.}\,]$ (you do not have to memorize this one)
	ARBITRARY-TERMED SERIES TESTS $\sum a_n$ where $-\infty < a_n < \infty \ \forall n \in \mathbb{N}$
Ratio Test	Let $\rho = \lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert$.
Root Test	Let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} \overset{\text{note}}{=} \lim_{n\to\infty} \lvert a_n\rvert^{\frac{1}{n}}$. $0 \leqslant \rho < 1 \Longrightarrow \sum a_n$ converges absolutely $\rho = 1 \Longrightarrow$ test is inconclusive $1 < \rho \leqslant \infty \Longrightarrow \sum a_n$ diverges (by n^{th} term test for divergence)
	ALTERNATING SERIES TEST $\sum a_n = \sum (-1)^n u_n$ where $u_n > 0\ \forall n \in \mathbb{N}$, in other words $a_n = (-1)^n u_n$ and $u_n > 0$
Alternating Series Test (AST)	$[\,u_n > u_{n+1}\ \forall n \in \mathbb{N}\ \&\ \lim_{n\to\infty} u_n = 0\,] \Longrightarrow [\,\sum a_n = \sum (-1)^n u_n \text{ conv.}\,]$