**Techniques of Integration** **Trigonometric Substitution** $\boxed{a > 0 \text{ throughout this concept}}$

Try using **Trig Substitution** for integrands involving: $\boxed{a^2 - u^2}$ or $\boxed{a^2 + u^2}$ or $\boxed{u^2 - a^2}$.

**SUMMARY CHART**

| | recall | integrand has | trig. sub. working form | trig. sub. reality form | restrictions on $u$ | restrictions on $\theta$ |
|---|---|---|---|---|---|---|
| (1) | $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$ | $a^2 - u^2$ | $u = a\sin\theta$ | $\theta = \arcsin\left(\frac{u}{a}\right)$ | $\left\lvert\frac{u}{a}\right\rvert \le 1$ | $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $(\star_1)$ |
| (2) | $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$ | $a^2 + u^2$ | $u = a\tan\theta$ | $\theta = \arctan\left(\frac{u}{a}\right)$ | none | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $(\star_2)$ |
| (3) | $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{\lvert u\rvert}{a} + c$ | $u^2 - a^2$ | $u = a\sec\theta$ | $\theta = \operatorname{arcsec}\left(\frac{u}{a}\right)$ | $1 \le \left\lvert\frac{u}{a}\right\rvert$ | |
| $(3^+)$ | if $u$ is positive | | | | $a \le u$ | $0 \le \theta < \frac{\pi}{2}$ $(\star_{3^+})$ |
| $(3^-)$ | if $u$ is negative | | | | $u \le -a$ | $\frac{\pi}{2} < \theta \le \pi$ $(\star_{3^-})$ |

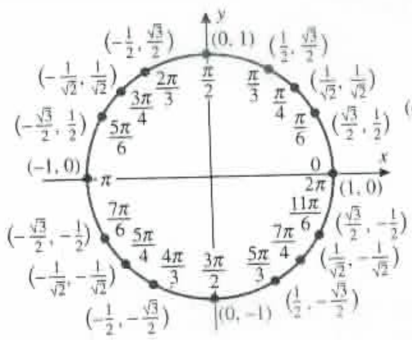
**USE.** This serves also as a *self-check* : if $(\triangle)$ does not work out, then you picked the wrong trig substitution.

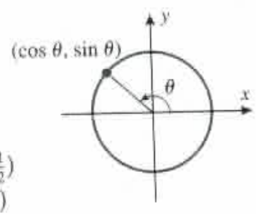
| | | |
|---|---|---|
| (1) | $\sqrt{a^2-u^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} \overset{(\triangle)}{=} a\lvert\cos\theta\rvert \overset{(\star_1)}{=} a\cos\theta$ | |
| (2) | $\sqrt{a^2+u^2} = \sqrt{a^2 + a^2\tan^2\theta} = a\sqrt{1+\tan^2\theta} \overset{(\triangle)}{=} a\lvert\sec\theta\rvert \overset{(\star_2)}{=} a\sec\theta$ | |
| $(3^+)$ | $\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} \overset{(\triangle)}{=} a\lvert\tan\theta\rvert \overset{(\star_{3^+})}{=} +a\tan\theta$ | If book does not provided enough info to determine |
| $(3^-)$ | $\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} \overset{(\triangle)}{=} a\lvert\tan\theta\rvert \overset{(\star_{3^-})}{=} -a\tan\theta$ | which (3) to use, then use $(3^+)$. |

**USEFUL:** Another self-check/memory-trick ... ¿¿ $u = a\sin\theta$ or $u = a\tan\theta$ or $u = a\sec\theta$ ??

| | know | try to write as: $\boxed{\pm u^2 \pm 1^2}$ = (a trig function)$(\theta)$ | so | now think of $1^2$ as $a^2$ |
|---|---|---|---|---|
| (1) | $\cos^2\theta + \sin^2\theta = 1$ | $\cos^2\theta = 1 - \sin^2\theta = \boxed{1^2 - u^2}$ let $u = 1\ \sin\theta$ | $\Rightarrow$ | $\boxed{a^2 - u^2} \rightsquigarrow u = a\sin\theta$ |
| (2) | $1 + \tan^2\theta = \sec^2\theta$ | $\sec^2\theta = 1 + \tan^2\theta = \boxed{1^2 + u^2}$ let $u = 1\ \tan\theta$ | $\Rightarrow$ | $\boxed{a^2 + u^2} \rightsquigarrow u = a\tan\theta$ |
| (3) | $1 + \tan^2\theta = \sec^2\theta$ | $\tan^2\theta = \sec^2\theta - 1 = \boxed{u^2 - 1^2}$ let $u = 1\ \sec\theta$ | $\Rightarrow$ | $\boxed{u^2 - a^2} \rightsquigarrow u = a\sec\theta$ |

**Unit Circle – Know Me !!**




**Review of Inverse Trig Functions**

| inverse trig function | other notation | domain | range | memory trick |
|---|---|---|---|---|
| $\alpha = \arcsin y$ | $\alpha = \sin^{-1} y$ | $-1 \le y \le 1$ | $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ | c o s<br>i<br>n |
| $\alpha = \arccos y$ | $\alpha = \cos^{-1} y$ | $-1 \le y \le 1$ | $0 \le \alpha \le \pi$ | |
| $\alpha = \arctan y$ | $\alpha = \tan^{-1} y$ | $-\infty < y < \infty$ | $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ | c o t<br>a<br>n |
| $\alpha = \text{arccot}\ y$ | $\alpha = \cot^{-1} y$ | $-\infty < y < \infty$ | $0 < \alpha < \pi$ | |
| $\alpha = \text{arcsec}\ y$ | $\alpha = \sec^{-1} y$ | $\|y\| \ge 1$ | $0 \le \alpha \le \pi$ , $\alpha \ne \frac{\pi}{2}$ | s e c<br>s<br>c |
| $\alpha = \text{arccsc}\ y$ | $\alpha = \csc^{-1} y$ | $\|y\| \ge 1$ | $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ , $\alpha \ne 0$ | |

**Useful Trig Identities**

| | | |
|---|---|---|
| $\cos^2\theta + \sin^2\theta = 1$ | $\xrightarrow{\text{divide through by } \cos^2\theta}$ | $1 + \tan^2\theta = \sec^2\theta$ |
| double angle formula: | $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ | $\sin(2\theta) = 2\sin\theta\cos\theta$ |