Telescoping Series

EXAMPLE 6 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

(see Section 7.4). Thus we have

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

■ Notice that the terms cancel in pairs. This is an example of a **telescoping sum**: Because of all the cancellations, the sum collapses (like a pirate's collapsing telescope) into just two terms.

and so
$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

□

■ Figure 3 illustrates Example 6 by showing the graphs of the sequence of terms $a_n = 1/[n(n+1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \to 0$ and $s_n \to 1$. See Exercises 62 and 63 for two geometric interpretations of Example 6.

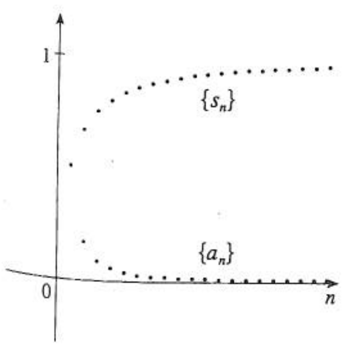

FIGURE 3

Ex 4 $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$ ← Not a geometric series but it is a Telescoping Series

$$\frac{2}{k(k+2)} \underset{=}{\overset{PFD}{=}} \frac{A}{k} + \frac{B}{k+2} \xRightarrow{\text{work}} \frac{1}{k} + \frac{-1}{k+2}$$

Look at partial sums S_N :

$$S_N = \sum_{k=1}^{N} \frac{2}{k(k+2)} = \sum_{k=1}^{N} \left[\frac{1}{k} + \frac{-1}{k+2} \right]$$

$$\left.\begin{aligned}
&= 1 + -\frac{1}{3} && \leftarrow k=1 \\
&\quad + \frac{1}{2} + \frac{-1}{4} && \leftarrow k=2 \\
&\quad + \frac{1}{3} + -\frac{1}{5} && \leftarrow k=3 \\
&\quad + \frac{1}{4} + -\frac{1}{6} && \leftarrow k=4 \\
&\quad \vdots \\
&\quad + \frac{1}{N} + \frac{-1}{N+2} && \leftarrow k=N
\end{aligned}\right\}$$

k & $k+2$ differ by $\underline{\underline{2}}$ so regroup in sets of 2's

Ex 4 — continued

$$
\begin{aligned}
S_N = {} & 1 + \cancel{\tfrac{-1}{3}} + \tfrac{1}{2} + \cancel{\tfrac{-1}{4}} && \leftarrow k = 1, 2 \\
& \quad ① \qquad\qquad ② \\
& + \cancel{\tfrac{1}{3}} + \cancel{\tfrac{-1}{5}} + \cancel{\tfrac{1}{4}} + \cancel{\tfrac{-1}{6}} && \leftarrow k = 3, 4 \\
& \quad ③ \qquad\qquad ④ \\
& + \cancel{\tfrac{1}{5}} + \cancel{\tfrac{-1}{7}} + \cancel{\tfrac{1}{6}} + \cancel{\tfrac{-1}{8}} && \leftarrow k = 5, 6 \\
& \quad ⑤ \qquad\qquad ⑥ \\
& + \cancel{\tfrac{1}{7}} + \cancel{\tfrac{-1}{9}} + \cancel{\tfrac{1}{8}} + \cancel{\tfrac{-1}{10}} && \leftarrow k = 7, 8 \\
& \quad ⑦ \qquad\qquad ⑧ \\
& \vdots \\
& + \cancel{\tfrac{1}{N-1}} + \tfrac{-1}{N+1} + \cancel{\tfrac{1}{N}} + \tfrac{-1}{N+2} && \leftarrow k = N-1, N
\end{aligned}
$$

$$\boxed{S_N = 1 + \frac{1}{2} + \frac{-1}{N+1} + \frac{-1}{N+2}} \xrightarrow{N \to \infty} 1 + \frac{1}{2} + 0 + 0 = \frac{3}{2}.$$

So $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$ converges and $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \boxed{\frac{3}{2}}$

Self Check

$$S_3 = \left(1 - \tfrac{1}{\underline{3}}\right) + \left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left(\tfrac{1}{\underline{3}} - \tfrac{1}{5}\right) = \frac{3}{2} - \frac{1}{3+1} + \frac{-1}{3+2}$$

$$S_4 = \left(1 - \tfrac{1}{\underline{3}}\right) + \left(\tfrac{1}{2} - \tfrac{1}{\underline{4}}\right) + \left(\tfrac{1}{\underline{3}} - \tfrac{1}{5}\right) + \left(\tfrac{1}{\underline{4}} + -\tfrac{1}{6}\right)$$

$$= \frac{3}{2} + \frac{-1}{4+1} + \frac{-1}{4+2}$$

etc...