SEQUENCES

A **sequence** $\{a_n\}_{n=1}^{\infty}$ is an ordered list of numbers. Thinks of as

$$\{a_n\}_{n=1}^{\infty} \; = \; \{a_1, a_2, a_3, a_4, \ldots\}$$

goes on forever $\Uparrow$.

You can also think of a sequence as a function $f\colon \mathbb{N} \to \mathbb{R}$ with $f(n) = a_n$.

Def 11.1.1: limit of a sequence (intuition). A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n \; = \; L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise we say the sequence **diverges** (or is **divergent**).

Def 11.1.2: limit of a sequence (precise). A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n \; = \; L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for each $\varepsilon > 0$ there is a corresponding integer N such that if $n > N$ then $|a_n - L| < \varepsilon$. In shorthand

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad [n > N \implies |a_n - L| < \varepsilon \,]$$

or equivalently

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad [n > N \implies L - \varepsilon < a_n < L + \varepsilon \,]$$

Fun Compare Def 11.1.2 (page 677, limit of a sequence) to Def 2.6.7 (page 138, limit of a function).

Def 2.6.7: limit of a function Let $f\colon (a, \infty) \to \mathbb{R}$ be a function. Then

$$\lim_{x\to\infty} f(x) \; = \; L$$

if for each $\varepsilon > 0$ there is a corresponding real number N such that if $x > N$ then $|f(x) - L| < \varepsilon$.

Theorem 11.1.3 If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for each integer n, then $\lim_{n\to\infty} a_n = L$. I.e.

$$\Big[\lim_{x\to\infty} f(x) = L\Big] \qquad \Longrightarrow \qquad \Big[\lim_{n\to\infty} f(n) = L\Big]$$

Warning:

$$\Big[\lim_{x\to\infty} f(x) = L\Big] \qquad \overset{\text{DOES NOT}}{\Longleftarrow} \qquad \Big[\lim_{n\to\infty} f(n) = L\Big]$$

Corollary 11.1.4 Theorem 2.6.5 says $\lim_{x\to\infty} \frac{1}{x^r} = 0$ when $r > 0$. So

$$\lim_{n\to\infty} \frac{1}{n^r} \; = \; 0 \qquad \text{if } r > 0 \, .$$

Def 11.1.5 $\lim_{n\to\infty} a_n = \infty$ means that for each positive (big) number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M \ .$$

In this case we say that the sequence $\{a_n\}$ **diverges** to infinity.

Limit Laws for Sequences Let $\{a_n\}$ and $\{b_n\}$ be **convergent** sequences. Let $c \in \mathbb{R}$ be a constant.

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} c = c$$
$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0 \ .$$

Squeeze Theorem for Sequences If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

Theorem 11.1.6 If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 11.1.7 If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. In short:

$$a_n \overset{n\to\infty}{\longrightarrow} L \qquad \overset{f \text{ continuous at } L}{\Longrightarrow} \qquad f(a_n) \overset{n\to\infty}{\longrightarrow} f(L) \ .$$

Theorem 11.1.9 Let r be some fixed number.

$$\lim_{n\to\infty} r^n = \begin{cases} \text{diverges} & \text{if } \ r \le -1 \\ 0 & \text{if } \ -1 < r < 1 \\ 1 & \text{if } \ r = 1 \\ \text{diverges to } \infty & \text{if } \ r > 1 \end{cases}$$

Def 11.1.10 Let $\{a_n\}$ be a sequence.
$\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$, that is $a_1 < a_2 < a_3 < \ldots$.
$\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$, that is $a_1 > a_2 > a_3 > \ldots$.
$\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

Def 11.1.11 A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \le M \ \text{ for all } \ n \ge 1 \ .$$

It is **bounded below** if there is a number m such that

$$m \le a_n \ \text{ for all } \ n \ge 1 \ .$$

If it is bounded above AND below, then its called a **bounded sequence**.

Theorem 11.1.12: Monotonic Sequence Theorem
If $\{a_n\}$ is increasing and bounded above, then $\{a_n\}$ converges.
If $\{a_n\}$ is decreasing and bounded below, then $\{a_n\}$ converges.