A sequence \( \{a_n\}_{n=1}^{\infty} \) is an ordered list of numbers. Thinks of as \( \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, a_4, \ldots\} \) goes on forever ↑.

You can also think of a sequence as a function \( f: \mathbb{N} \rightarrow \mathbb{R} \) with \( f(n) = a_n \).

**Def 11.1.1:** limit of a sequence (intuition). A sequence \( \{a_n\} \) has the limit \( L \) and we write

\[
\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty
\]

if we can make the terms \( a_n \) as close to \( L \) as we like by taking \( n \) sufficiently large. If \( \lim_{n \to \infty} a_n \) exists, we say the sequence converges (or is convergent). Otherwise we say the sequence diverges (or is divergent).

**Def 11.1.2:** limit of a sequence (precise). A sequence \( \{a_n\} \) has the limit \( L \) and we write

\[
\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty
\]

if for each \( \varepsilon > 0 \) there is a corresponding integer \( N \) such that if \( n > N \) then \(|a_n - L| < \varepsilon\). In shorthand

\[
\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad [n > N \implies |a_n - L| < \varepsilon]
\]

or equivalently

\[
\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad [n > N \implies L - \varepsilon < a_n < L + \varepsilon]
\]

**Fun** Compare Def 11.1.2 (page 677, limit of a sequence) to Def 2.6.7 (page 138, limit of a function).

**Def 2.6.7:** limit of a function Let \( f: (a, \infty) \rightarrow \mathbb{R} \) be a function. Then

\[
\lim_{x \to \infty} f(x) = L
\]

if for each \( \varepsilon > 0 \) there is a corresponding real number \( N \) such that if \( x > N \) then \(|f(x) - L| < \varepsilon\).

**Theorem 11.1.3** If \( \lim_{x \to \infty} f(x) = L \) and \( f(n) = a_n \) for each integer \( n \), then \( \lim_{n \to \infty} a_n = L \). I.e.

\[
\left[ \lim_{x \to \infty} f(x) = L \right] \quad \implies \quad \left[ \lim_{n \to \infty} f(n) = L \right]
\]

**Warning:**

\[
\left[ \lim_{x \to \infty} f(x) = L \right] \quad \text{DOES NOT} \quad \iff \quad \left[ \lim_{n \to \infty} f(n) = L \right]
\]

**Corollary 11.1.4** Theorem 2.6.5 says \( \lim_{x \to \infty} \frac{1}{x^r} = 0 \) when \( r > 0 \). So

\[
\lim_{n \to \infty} \frac{1}{n^r} = 0 \quad \text{if} \quad r > 0
\]
Def 11.1.5 $\lim_{n \to \infty} a_n = \infty$ means that for each positive (big) number $M$ there is an integer $N$ such that 

$$\text{if } n > N \text{ then } a_n > M.$$ 

In this case we say that the sequence $\{a_n\}$ diverges to infinity.

Limit Laws for Sequences Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Let $c \in \mathbb{R}$ be a constant.

$$\begin{align*}
\lim_{n \to \infty} (a_n + b_n) &= \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \\
\lim_{n \to \infty} (a_n - b_n) &= \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n \\
\lim_{n \to \infty} ca_n &= c \lim_{n \to \infty} a_n \\
\lim_{n \to \infty} (a_n b_n) &= \left( \lim_{n \to \infty} a_n \right) \cdot \left( \lim_{n \to \infty} b_n \right) \\
\lim_{n \to \infty} \frac{a_n}{b_n} &= \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \text{ if } \lim_{n \to \infty} b_n \neq 0 \\
\lim_{n \to \infty} a_n^p &= \left( \lim_{n \to \infty} a_n \right)^p \text{ if } p > 0 \text{ and } a_n > 0.
\end{align*}$$

Squeeze Theorem for Sequences If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n$, then $\lim_{n \to \infty} b_n = L$.

Theorem 11.1.6 If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

Theorem 11.1.7 If $\lim_{n \to \infty} a_n = L$ and the function $f$ is continuous at $L$, then $\lim_{n \to \infty} f(a_n) = f(L)$. In short:

$$a_n \xrightarrow{n \to \infty} L \quad f \text{ continuous at } L \quad f(a_n) \xrightarrow{n \to \infty} f(L).$$

Theorem 11.1.9 Let $r$ be some fixed number.

$$\lim_{n \to \infty} r^n = \begin{cases} 
\text{diverges} & \text{if } r \leq -1 \\
0 & \text{if } -1 < r < 1 \\
1 & \text{if } r = 1 \\
\text{diverges to } \infty & \text{if } r > 1
\end{cases}$$

Def 11.1.10 Let $\{a_n\}$ be a sequence.

$\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < a_3 < \ldots$.

$\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$, that is $a_1 > a_2 > a_3 > \ldots$.

$\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

Def 11.1.11 A sequence $\{a_n\}$ is **bounded above** if there is a number $M$ such that

$$a_n \leq M \text{ for all } n \geq 1.$$

It is **bounded below** if there is a number $m$ such that

$$m \leq a_n \text{ for all } n \geq 1.$$

If it is bounded above AND below, then its called a **bounded sequence**.

Theorem 11.1.12: Monotonic Sequence Theorem

If $\{a_n\}$ is increasing and bounded above, then $\{a_n\}$ converges.

If $\{a_n\}$ is decreasing and bounded below, then $\{a_n\}$ converges.