INTEGRAL TEST

RECALL

If $\{x_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence (i.e., $x_n \leq x_{n+1}$), then
either

- $\{x_n\}_{n\in\mathbb{N}}$ is bounded above, in which case $\lim_{n\to\infty} x_n$ exists

or

- $\{x_n\}_{n\in\mathbb{N}}$ is not bounded above, in which case $\lim_{n\to\infty} x_n = \infty$.

POSITIVE-TERM SERIES

Definition: $\sum a_n$ is a positive-term series if $a_n \geq 0$ for each n.

Explore: Let $\sum a_n$ be a positive-term series.

(1) Consider its *sequence* of partial sums $\{S_N\}_{N\in\mathbb{N}}$ where $S_N = a_1 + a_2 + \ldots + a_N$.

(2) Recall that $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$.

(3) $0 \leq a_n$ for each $n \in \mathbb{N}$ $\implies$ $S_N \leq S_{N+1}$ for each $N \in \mathbb{N}$.

(4) So $\{S_N\}_{N\in\mathbb{N}}$ is a non-decreasing sequence.

So either:

- $\{S_N\}_{N\in\mathbb{N}}$ is bounded above and so $\lim_{N\to\infty} S_N$ exists and so $\sum a_n$ converges

or

- $\{S_N\}_{N\in\mathbb{N}}$ is not bounded above and so $\lim_{N\to\infty} S_N = \infty = \sum_{n=1}^{\infty} a_n$ and so $\sum a_n$ diverges.

TODAY'S GOAL

Examine the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \rightsquigarrow \quad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \ . \end{cases}$$

When $p = 1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ is also called the harmonic series.

- Let $\sum a_n$ be a positive-term series.
- Find a function $f(x)$ such that

  (1) $f(n) = a_n$ for $n \in \mathbb{N}$

  (2) f is decreasing for $x \geq 1$ (often check this by showing $f' < 0$)

  (3) f is continuous for $x \geq 1$

  (4) so $f(x) \geq 0$ for $x \geq 1$.

Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ either:

(1) both converge (to different numbers most likely)

(2) both diverge .

This is because $\{\sum_{n=1}^{N} a_n\}_{N \in \mathbb{N}}$ and $\{\int_1^N f(x)\,dx\}_{N \in \mathbb{N}}$ are both non-decreasing sequences and so each has the choice of either (converging to some finite number) <u>or</u> (diverging to ∞).

But

$$a_2 + a_3 + \ldots + a_N \quad \leq \quad \int_1^N f(x)\,dx \quad \leq \quad a_1 + a_2 + \ldots + a_{N-1}$$

Now take the limit as $N \to \infty$ to see that

$$\sum_{n=2}^{\infty} a_n \quad \leq \quad \int_1^{\infty} f(x)\,dx \quad \leq \quad \sum_{n=1}^{\infty} a_n \ .$$