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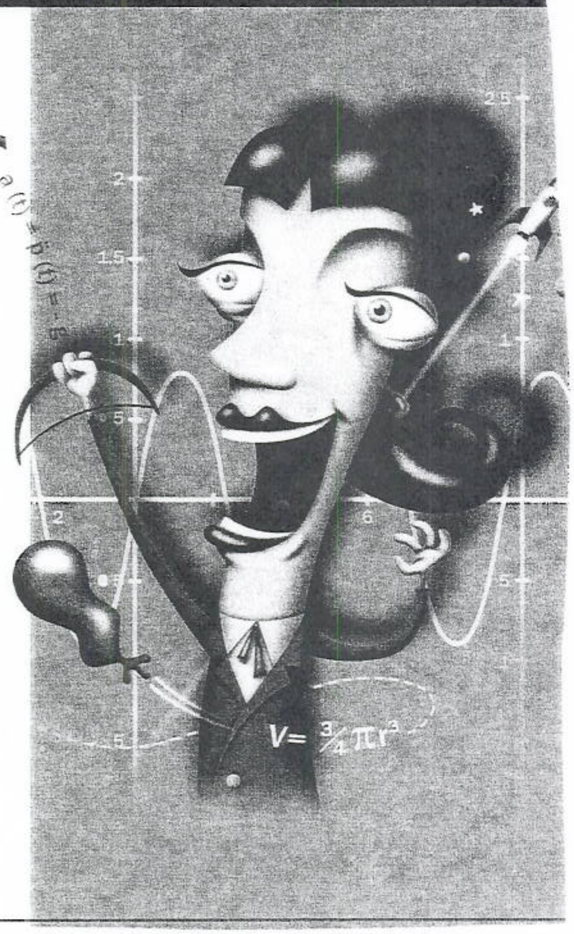

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CHAPTER 5

Indeterminate Forms

5.1 l'Hôpital's Rule

5.1.1 INTRODUCTION

Consider the limit

$$\lim_{x \to c} \frac{f(x)}{g(x)}. \qquad (*)$$

If $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} g(x)$ exists and is not zero then the limit $(*)$ is straightforward to evaluate. However, as we saw in Theorem 2.3, when $\lim_{x \to c} g(x) = 0$ then the situation is more complicated (especially when $\lim_{x \to c} f(x) = 0$ as well).

For example, if $f(x) = \sin x$ and $g(x) = x$ then the limit of the quotient as $x \to 0$ exists and equals 1. However if $f(x) = x$ and $g(x) = x^2$ then the limit of the quotient as $x \to 0$ does not exist.

In this section we learn a rule for evaluating indeterminate forms of the type $(*)$ when either $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty$. Such limits, or "forms," are considered indeterminate because the limit of the quotient might actually exist and be finite or might not exist; one cannot analyze such a form by elementary means.

5.1.2 L'HÔPITAL'S RULE

Theorem 5.1 **(l'Hôpital's Rule)**
Let $f(x)$ and $g(x)$ be differentiable functions on $(a, c) \cup (c, b)$. If

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided this last limit exists as a finite or infinite limit.

Let us learn how to use this new result.

EXAMPLE 5.1

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x^2 + x - 2}.$$

SOLUTION

We first notice that both the numerator and denominator have limit zero as x tends to 1. Thus the quotient is indeterminate at 1 and of the form 0/0. l'Hôpital's Rule therefore applies and the limit equals

$$\lim_{x \to 1} \frac{(d/dx)(\ln x)}{(d/dx)(x^2 + x - 2)},$$

provided this last limit exists. The last limit is

$$\lim_{x \to 1} \frac{1/x}{2x + 1} = \lim_{x \to 1} \frac{1}{2x^2 + x}.$$

Therefore we see that

$$\lim_{x \to 1} \frac{\ln x}{x^2 + x - 2} = \frac{1}{3}.$$

You Try It: Apply l'Hôpital's Rule to the limit $\lim_{x \to 2} \sin(\pi x)/(x^2 - 4)$.

You Try It: Use l'Hôpital's Rule to evaluate $\lim_{h \to 0}(\sin h/h)$ and $\lim_{h \to 0}(\cos h - 1/h)$. These limits are important in the theory of calculus.

EXAMPLE 5.2

Evaluate the limit

$$\lim_{x \to 0} \frac{x^3}{x - \sin x}.$$

SOLUTION

As $x \to 0$ both numerator and denominator tend to zero, so the quotient is indeterminate at 0 of the form $0/0$. Thus l'Hôpital's Rule applies. Our limit equals

$$\lim_{x\to 0} \frac{(d/dx)x^3}{(d/dx)(x - \sin x)},$$

provided that this last limit exists. It equals

$$\lim_{x\to 0} \frac{3x^2}{1 - \cos x}.$$

This is another indeterminate form. So we must again apply l'Hôpital's Rule. The result is

$$\lim_{x\to 0} \frac{6x}{\sin x}.$$

This is again indeterminate; another application of l'Hôpital's Rule gives us finally

$$\lim_{x\to 0} \frac{6}{\cos x} = 6.$$

We conclude that the original limit equals 6.

You Try It: Apply l'Hôpital's Rule to the limit $\lim_{x\to 0} x/[1/\ln|x|]$.

Indeterminate Forms Involving ∞ We handle indeterminate forms involving infinity as follows: Let $f(x)$ and $g(x)$ be differentiable functions on $(a, c) \cup (c, b)$. If

$$\lim_{x\to c} f(x) \text{ and } \lim_{x\to c} g(x)$$

both exist and equal $+\infty$ or $-\infty$ (they may have the same sign or different signs) then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

provided this last limit exists either as a finite or infinite limit.

Let us look at some examples.

EXAMPLE 5.3

Evaluate the limit

$$\lim_{x\to 0} x^3 \cdot \ln|x|.$$

SOLUTION

This may be rewritten as

$$\lim_{x \to 0} \frac{\ln |x|}{1/x^3}.$$

Notice that the numerator tends to $-\infty$ and the denominator tends to $\pm\infty$ as $x \to 0$. Thus the quotient is indeterminate at 0 of the form $-\infty/+\infty$. So we may apply l'Hôpital's Rule for infinite limits to see that the limit equals

$$\lim_{x \to 0} \frac{1/x}{-3x^{-4}} = \lim_{x \to 0} -x^3/3 = 0.$$

Yet another version of l'Hôpital's Rule, this time for unbounded intervals, is this: Let f and g be differentiable functions on an interval of the form $[A, +\infty)$. If $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = 0$ or if $\lim_{x \to +\infty} f(x) = \pm\infty$ and $\lim_{x \to +\infty} g(x) = \pm\infty$, then

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}$$

provided that this last limit exists either as a finite or infinite limit. The same result holds for f and g defined on an interval of the form $(-\infty, B]$ and for the limit as $x \to -\infty$.

EXAMPLE 5.4

Evaluate

$$\lim_{x \to +\infty} \frac{x^4}{e^x}.$$

SOLUTION

We first notice that both the numerator and the denominator tend to $+\infty$ as $x \to +\infty$. Thus the quotient is indeterminate at $+\infty$ of the form $+\infty/+\infty$. Therefore the new version of l'Hôpital's Rule applies and our limit equals

$$\lim_{x \to +\infty} \frac{4x^3}{e^x}.$$

Again the numerator and denominator tend to $+\infty$ as $x \to +\infty$, so we once more apply l'Hôpital's Rule. The limit equals

$$\lim_{x \to +\infty} \frac{12x^2}{e^x} = 0.$$

We must apply l'Hôpital's Rule two more times. We first obtain

$$\lim_{x \to +\infty} \frac{24x}{e^x}$$

and then

$$\lim_{x \to +\infty} \frac{24}{e^x}.$$

We conclude that

$$\lim_{x \to +\infty} \frac{x^4}{e^x} = 0.$$

You Try It: Evaluate the limit $\lim_{x \to +\infty} \left(\frac{e^x}{x \ln x} \right)$.

You Try It: Evaluate the limit $\lim_{x \to -\infty} x^4 \cdot e^x$.

EXAMPLE 5.5

Evaluate the limit

$$\lim_{x \to -\infty} \frac{\sin(2/x)}{\sin(5/x)}.$$

SOLUTION

We note that both numerator and denominator tend to 0, so the quotient is indeterminate at $-\infty$ of the form $0/0$. We may therefore apply l'Hôpital's Rule. Our limit equals

$$\lim_{x \to -\infty} \frac{(-2/x^2)\cos(2/x)}{(-5/x^2)\cos(5/x)}.$$

This in turn simplifies to

$$\lim_{x \to -\infty} \frac{2\cos(2/x)}{5\cos(5/x)} = \frac{2}{5}.$$

l'Hôpital's Rule also applies to one-sided limits. Here is an example.

EXAMPLE 5.6

Evaluate the limit

$$\lim_{x \to 0^+} \frac{\sin \sqrt{x}}{\sqrt{x}}.$$

SOLUTION

Both numerator and denominator tend to zero so the quotient is indeterminate at 0 of the form $0/0$. We may apply l'Hôpital's Rule; differentiating numerator and denominator, we find that the limit equals

$$\lim_{x \to 0^+} \frac{[\cos \sqrt{x}] \cdot (1/2)x^{-1/2}}{(1/2)x^{-1/2}} = \lim_{x \to 0^+} \cos \sqrt{x}$$
$$= 1.$$

You Try It: How can we apply l'Hôpital's Rule to evaluate $\lim_{x \to 0^+} x \cdot \ln x$?

5.2 Other Indeterminate Forms

5.2.1 INTRODUCTION

By using some algebraic manipulations, we can reduce a variety of indeterminate limits to expressions which can be treated by l'Hôpital's Rule. We explore some of these techniques in this section.

5.2.2 WRITING A PRODUCT AS A QUOTIENT

The technique of the first example is a simple one, but it is used frequently.

EXAMPLE 5.7

Evaluate the limit

$$\lim_{x \to -\infty} x^2 \cdot e^{3x}.$$

SOLUTION

Notice that $x^2 \to +\infty$ while $e^{3x} \to 0$. So the limit is indeterminate of the form $0 \cdot \infty$. We rewrite the limit as

$$\lim_{x \to -\infty} \frac{x^2}{e^{-3x}}.$$

Now both numerator and denominator tend to infinity and we may apply l'Hôpital's Rule. The result is that the limit equals

$$\lim_{x \to -\infty} \frac{2x}{-3e^{-3x}}.$$

Again the numerator and denominator both tend to infinity so we apply l'Hôpital's Rule to obtain:

$$\lim_{x \to -\infty} \frac{2}{9e^{-3x}}.$$

It is clear that the limit of this last expression is zero. We conclude that

$$\lim_{x \to -\infty} x \cdot e^{3x} = 0.$$

You Try It: Evaluate the limit $\lim_{x \to +\infty} e^{-\sqrt{x}} \cdot x$.

5.2.3 THE USE OF THE LOGARITHM

The natural logarithm can be used to reduce an expression involving exponentials to one involving a product or a quotient.

EXAMPLE 5.8

Evaluate the limit

$$\lim_{x \to 0^+} x^x.$$

SOLUTION

We study the limit of $f(x) = x^x$ by considering $\ln f(x) = x \cdot \ln x$. We rewrite this as

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln x}{1/x}.$$

Both numerator and denominator tend to $\pm\infty$, so the quotient is indeterminate of the form $-\infty/\infty$. Thus l'Hôpital's Rule applies. The limit equals

$$\lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0.$$

Now the only way that $\ln f(x)$ can tend to zero is if $f(x) = x^x$ tends to 1. We conclude that

$$\lim_{x \to 0^+} x^x = 1.$$

EXAMPLE 5.9

Evaluate the limit

$$\lim_{x \to 0} (1 + x^2)^{\ln |x|}.$$

SOLUTION

Let $f(x) = (1 + x^2)^{\ln |x|}$ and consider $\ln f(x) = \ln |x| \cdot \ln(1 + x^2)$. This expression is indeterminate of the form $-\infty \cdot 0$.

We rewrite it as

$$\lim_{x \to 0} \frac{\ln(1 + x^2)}{1/\ln |x|},$$

so that both the numerator and denominator tend to 0. So l'Hôpital's Rule applies and we have

$$\lim_{x \to 0} \ln f(x) = \lim_{x \to 0} \frac{2x/(1 + x^2)}{-1/[x \ln^2(|x|)]} = \lim_{x \to 0} -\frac{2x^2 \ln^2(|x|)}{(1 + x^2)}.$$

The numerator tends to 0 (see Example 5.3) and the denominator tends to 1. Thus

$$\lim_{x \to 0} \ln f(x) = 0.$$

But the only way that $\ln f(x)$ can tend to zero is if $f(x)$ tends to 1. We conclude that

$$\lim_{x\to 0}(1+x^2)^{\ln|x|} = 1.$$

You Try It: Evaluate the limit $\lim_{x\to 0^+}(1/x)^x$.

You Try It: Evaluate the limit $\lim_{x\to 0^+}(1+x)^{1/x}$. In fact this limit gives an important way to define Euler's constant e (see Sections 1.9 and 6.2.3).

5.2.4 PUTTING TERMS OVER A COMMON DENOMINATOR

Many times a simple algebraic manipulation involving fractions will put a limit into a form which can be studied using l'Hôpital's Rule.

EXAMPLE 5.10

Evaluate the limit

$$\lim_{x\to 0}\left[\frac{1}{\sin 3x} - \frac{1}{3x}\right].$$

SOLUTION

We put the fractions over a common denominator to rewrite our limit as

$$\lim_{x\to 0}\left[\frac{3x - \sin 3x}{3x\cdot\sin 3x}\right].$$

Both numerator and denominator vanish as $x \to 0$. Thus the quotient has indeterminate form $0/0$. By l'Hôpital's Rule, the limit is therefore equal to

$$\lim_{x\to 0}\frac{3 - 3\cos 3x}{3\sin 3x + 9x\cos 3x}.$$

This quotient is still indeterminate; we apply l'Hôpital's Rule again to obtain

$$\lim_{x\to 0}\frac{9\sin 3x}{18\cos 3x - 27x\sin 3x} = 0.$$

EXAMPLE 5.11

Evaluate the limit

$$\lim_{x\to 0}\left[\frac{1}{4x} - \frac{1}{e^{4x}-1}\right].$$

SOLUTION

The expression is indeterminate of the form $\infty - \infty$. We put the two fractions over a common denominator to obtain

$$\lim_{x \to 0} \frac{e^{4x} - 1 - 4x}{4x(e^{4x} - 1)}.$$

Notice that the numerator and denominator both tend to zero as $x \to 0$, so this is indeterminate of the form $0/0$. Therefore l'Hôpital's Rule applies and our limit equals

$$\lim_{x \to 0} \frac{4e^{4x} - 4}{4e^{4x}(1 + 4x) - 4}.$$

Again the numerator and denominator tend to zero and we apply l'Hôpital's Rule; the limit equals

$$\lim_{x \to 0} \frac{16e^{4x}}{16e^{4x}(2 + 4x)} = \frac{1}{2}.$$

You Try It: Evaluate the limit $\lim_{x \to 0} \left(\frac{1}{\cos x - 1} \right) + \left(\frac{2}{x^2} \right)$.

5.2.5 OTHER ALGEBRAIC MANIPULATIONS

Sometimes a factorization helps to clarify a subtle limit:

EXAMPLE 5.12

Evaluate the limit

$$\lim_{x \to +\infty} \left[x^2 - (x^4 + 4x^2 + 5)^{1/2} \right].$$

SOLUTION

The limit as written is of the form $\infty - \infty$. We rewrite it as

$$\lim_{x \to +\infty} x^2 \left[1 - (1 + 4x^{-2} + 5x^{-4})^{1/2} \right] = \lim_{x \to +\infty} \frac{1 - (1 + 4x^{-2} + 5x^{-4})^{1/2}}{x^{-2}}.$$

Notice that both the numerator and denominator tend to zero, so it is now indeterminate of the form $0/0$. We may thus apply l'Hôpital's Rule. The result is that the limit equals

$$\lim_{x \to +\infty} \frac{(-1/2)(1 + 4x^{-2} + 5x^{-4})^{-1/2} \cdot (-8x^{-3} - 20x^{-5})}{-2x^{-3}}$$

$$= \lim_{x \to +\infty} -(1 + 4x^{-2} + 5x^{-4})^{-1/2} \cdot (2 + 5x^{-2}).$$

Since this last limit is -2, we conclude that

$$\lim_{x\to+\infty}\left[x^2-(x^4+4x^2+5)^{1/2}\right]=-2.$$

EXAMPLE 5.13

Evaluate

$$\lim_{x\to-\infty}\left[e^{-x}-(e^{-3x}-x^4)^{1/3}\right].$$

SOLUTION

First rewrite the limit as

$$\lim_{x\to-\infty} e^{-x}\left[1-(1-x^4e^{3x})^{1/3}\right]=\lim_{x\to-\infty}\frac{1-(1-x^4e^{3x})^{1/3}}{e^x}.$$

Notice that both the numerator and denominator tend to zero (here we use the result analogous to Example 5.7 that $x^4e^{3x}\to 0$). So our new expression is indeterminate of the form $0/0$. l'Hôpital's Rule applies and our limit equals

$$\lim_{x\to-\infty}\frac{-(1/3)(1-x^4e^{3x})^{-2/3}\cdot(-4x^3\cdot e^{3x}-x^4\cdot 3e^{3x})}{e^x}$$
$$=\lim_{x\to-\infty}(1/3)(1-x^4e^{3x})^{-2/3}(4x^3\cdot e^{2x}+3x^4\cdot e^{2x}).$$

Just as in Example 5.7, $x^4\cdot e^{3x}x^3\cdot e^{2x}$, and $x^4\cdot e^{2x}$ all tend to zero. We conclude that our limit equals 0.

You Try It: Evaluate $\lim_{x\to+\infty}[\sqrt{x+1}-\sqrt{x}]$.

5.3 Improper Integrals: A First Look

5.3.1 INTRODUCTION

The theory of the integral that we learned earlier enables us to integrate a continuous function $f(x)$ on a closed, bounded interval $[a, b]$. See Fig. 5.1. However, it is frequently convenient to be able to integrate an unbounded function, or a function defined on an unbounded interval. In this section and the next we learn to do so, and we see some applications of this new technique. The basic idea is that the integral of an unbounded function is the limit of integrals of bounded functions; likewise, the integral of a function on an unbounded interval is the limit of the integral on bounded intervals.

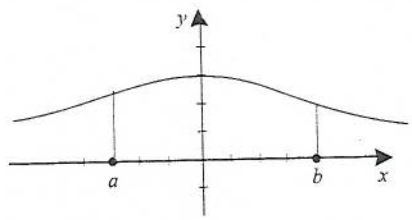

Fig. 5.1

5.3.2 INTEGRALS WITH INFINITE INTEGRANDS

Let f be a continuous function on the interval $[a, b)$ which is unbounded as $x \to b^-$ (see Fig. 5.2). The integral

$$\int_a^b f(x)\,dx$$

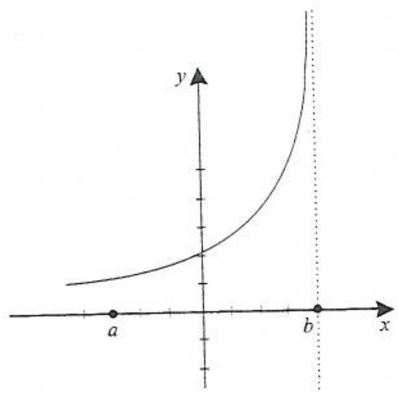

Fig. 5.2

is then called an *improper integral with infinite integrand* at b. We often just say "improper integral" because the *source* of the improperness will usually be clear from context. The next definition tells us how such an integral is evaluated.

If

$$\int_a^b f(x)\,dx$$

is an improper integral with infinite integrand at b then the value of the integral is defined to be

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx,$$

provided that this limit exists. See Fig. 5.3.

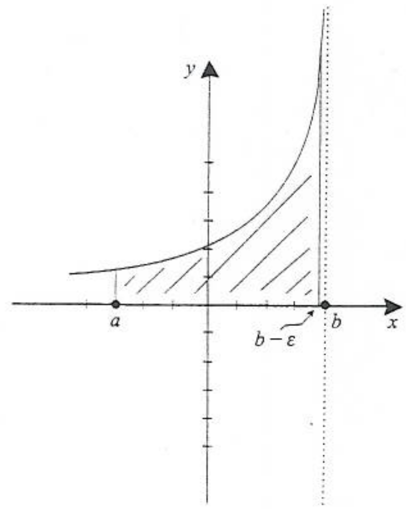

Fig. 5.3

EXAMPLE 5.14

Evaluate the integral

$$\int_2^8 4(8-x)^{-1/3}\,dx.$$

SOLUTION

The integral

$$\int_2^8 4(8-x)^{-1/3}\,dx$$

is an improper integral with infinite integrand at 8. According to the definition, the value of this integral is

$$\lim_{\epsilon\to 0^+}\int_2^{8-\epsilon} 4(8-x)^{-1/3}\,dx,$$

provided the limit exists. Since the integrand is continuous on the interval $[2, 8-\epsilon]$, we may calculate this last integral directly. We have

$$\lim_{\epsilon\to 0^+}\left[-6(8-x)^{2/3}\right]\Big|_2^{8-\epsilon} = \lim_{\epsilon\to 0^+} -6\left[\epsilon^{2/3}-6^{2/3}\right].$$

This limit is easy to evaluate: it equals $6^{5/3}$. We conclude that the integral is convergent and

$$\int_2^8 4(8-x)^{-1/3}\,dx = 6^{5/3}.$$

EXAMPLE 5.15

Analyze the integral

$$\int_2^3 (x-3)^{-2}\, dx.$$

SOLUTION

This is an improper integral with infinite integrand at 3. We evaluate this integral by considering

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_2^{3-\epsilon}(x-3)^{-2}\,dx &= \lim_{\epsilon\to 0^+} -(x-3)^{-1}\Big|_2^{3-\epsilon}\\ &= \lim_{\epsilon\to 0^+}\left[\epsilon^{-1}-1^{-1}\right].\end{aligned}$$

This last limit is $+\infty$. We therefore conclude that the improper integral *diverges*.

You Try It: Evaluate the improper integral $\int_{-2}^{-1}(dx/(x+1)^{4/5})\,dx$.

Improper integrals with integrand which is infinite at the left endpoint of integration are handled in a manner similar to the right endpoint case:

EXAMPLE 5.16

Evaluate the integral

$$\int_0^{1/2}\frac{1}{x\cdot \ln^2 x}\,dx.$$

SOLUTION

This integral is improper with infinite integrand at 0. The value of the integral is defined to be

$$\lim_{\epsilon\to 0^+}\int_\epsilon^{1/2}\frac{1}{x\cdot\ln^2 x}\,dx,$$

provided that this limit exists.

Since $1/(x\ln^2 x)$ is continuous on the interval $[\epsilon, 1/2]$ for $\epsilon > 0$, this last integral can be evaluated directly and will have a finite real value. For clarity, write $\varphi(x) = \ln x$, $\varphi'(x) = 1/x$. Then the (indefinite) integral becomes

$$\int \frac{\varphi'(x)}{\varphi^2(x)}\,dx.$$

Clearly the antiderivative is $-1/\varphi(x)$. Thus we see that

$$\lim_{\epsilon\to 0^+}\int_\epsilon^{1/2}\frac{1}{x\cdot\ln^2 x}\,dx = \lim_{\epsilon\to 0^+} -\frac{1}{\ln x}\Big|_\epsilon^{1/2} = \lim_{\epsilon\to 0^+}\left(\left[-\frac{1}{\ln(1/2)}\right]-\left[-\frac{1}{\ln\epsilon}\right]\right).$$

Now as $\epsilon \to 0^+$ we have $\ln \epsilon \to -\infty$ hence $1/\ln \epsilon \to 0$. We conclude that the improper integral *converges* to $1/\ln 2$.

You Try It: Evaluate the improper integral $\int_{-2}^{0} 1/(x+2)^{-1/2}\, dx$.

Many times the integrand has a singularity in the middle of the interval of integration. In these circumstances we divide the integral into two pieces for each of which the integrand is infinite at one endpoint, and evaluate each piece separately.

EXAMPLE 5.17

Evaluate the improper integral

$$\int_{-4}^{4} 4(x+1)^{-1/5}\, dx.$$

SOLUTION

The integrand is unbounded as x tends to -1. Therefore we evaluate separately the two improper integrals

$$\int_{-4}^{-1} 4(x+1)^{-1/5}\, dx \quad \text{and} \quad \int_{-1}^{4} 4(x+1)^{-1/5}\, dx.$$

The first of these has the value

$$\begin{aligned} \lim_{\epsilon \to 0^+} \int_{-4}^{-1-\epsilon} 4(x+1)^{-1/5}\, dx &= \lim_{\epsilon \to 0^+} 5(x+1)^{4/5}\Big|_{-4}^{-1-\epsilon} \\ &= \lim_{\epsilon \to 0^+} 5\{(-\epsilon)^{4/5} - (-3)^{4/5}\} \\ &= -5 \cdot 3^{4/5} \end{aligned}$$

The second integral has the value

$$\begin{aligned} \lim_{\epsilon \to 0^+} \int_{-1+\epsilon}^{4} 4(x+1)^{-1/5}\, dx &= \lim_{\epsilon \to 0^+} 5(x+1)^{4/5}\Big|_{-1+\epsilon}^{4} \\ &= \lim_{\epsilon \to 0^+} 5\{5^{4/5} - \epsilon^{4/5}\} \\ &= 5^{9/5}. \end{aligned}$$

We conclude that the original integral converges and

$$\begin{aligned} \int_{-4}^{4} 4(x+1)^{-1/5}\, dx &= \int_{-4}^{-1} 4(x+1)^{-1/5}\, dx + \int_{-1}^{4} 4(x+1)^{-1/5}\, dx \\ &= -5 \cdot 3^{4/5} + 5^{9/5}. \end{aligned}$$

You Try It: Evaluate the improper integral $\int_{-4}^{3} x^{-1}\, dx$.

It is dangerous to try to save work by not dividing the integral at the singularity. The next example illustrates what can go wrong.

EXAMPLE 5.18

Evaluate the improper integral

$$\int_{-2}^{2} x^{-4}\, dx.$$

SOLUTION

What we *should* do is divide this problem into the two integrals

$$\int_{-2}^{0} x^{-4}\, dx \quad \text{and} \quad \int_{0}^{2} x^{-4}\, dx. \qquad (*)$$

Suppose that instead we try to save work and just antidifferentiate:

$$\int_{-2}^{2} x^{-4}\, dx = -\frac{1}{3}x^{-3}\Big|_{-2}^{2} = -\frac{1}{12}.$$

A glance at Fig. 5.4 shows that something is wrong. The function x^{-4} is positive hence its integral should be positive too. However, since we used an incorrect method, we got a negative answer.

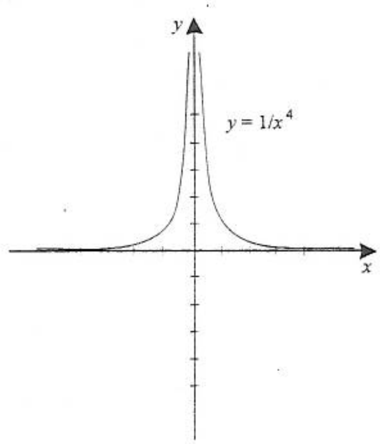

Fig. 5.4

In fact each of the integrals in line (∗) diverges, so *by definition* the improper integral

$$\int_{-2}^{2} x^{-4}\, dx$$

diverges.

EXAMPLE 5.19

Analyze the integral

$$\int_0^1 \frac{1}{x(1-x)^{1/2}}\,dx.$$

SOLUTION

The key idea is that we can only handle one singularity at a time. This integrand is singular at both endpoints 0 and 1. Therefore we divide the domain of integration somewhere in the middle—at $1/2$ say (it does not really matter where we divide)—and treat the two singularities separately.

First we treat the integral

$$\int_0^{1/2} \frac{1}{x(1-x)^{1/2}}\,dx.$$

Since the integrand has a singularity at 0, we consider

$$\lim_{\epsilon\to 0^+} \int_\epsilon^{1/2} \frac{1}{x(1-x)^{1/2}}\,dx.$$

This is a tricky integral to evaluate directly. But notice that

$$\frac{1}{x(1-x)^{1/2}} \geq \frac{1}{x\cdot(1)^{1/2}}$$

when $0 < \epsilon \leq x \leq 1/2$. Thus

$$\int_\epsilon^{1/2} \frac{1}{x(1-x)^{1/2}}\,dx \geq \int_\epsilon^{1/2} \frac{1}{x\cdot(1)^{1/2}}\,dx = \int_\epsilon^{1/2} \frac{1}{x}\,dx.$$

We evaluate the integral: it equals $\ln(1/2) - \ln \epsilon$. Finally,

$$\lim_{\epsilon\to 0^+} -\ln\epsilon = +\infty.$$

The first of our integrals therefore diverges.

But the full integral

$$\int_0^1 \frac{1}{x(1-x)^{1/2}}\,dx$$

converges if and only if each of the component integrals

$$\int_0^{1/2} \frac{1}{x(1-x)^{1/2}}\,dx$$

and

$$\int_{1/2}^1 \frac{1}{x(1-x)^{1/2}}\,dx$$

converges. Since the first integral diverges, we conclude that the original integral diverges as well.

You Try It: Calculate $\int_{-2}^{3}(2x)^{-1/3}\,dx$ as an improper integral.

5.3.3 AN APPLICATION TO AREA

Suppose that f is a non-negative, continuous function on the interval $(a, b]$ which is unbounded as $x \to a^+$. Look at Fig. 5.5. Let us consider the area under the graph of f and above the x-axis over the interval $(a, b]$. The area of the part of the region over the interval $[a+\epsilon, b]$, $\epsilon > 0$, is

$$\int_{a+\epsilon}^{b} f(x)\,dx.$$

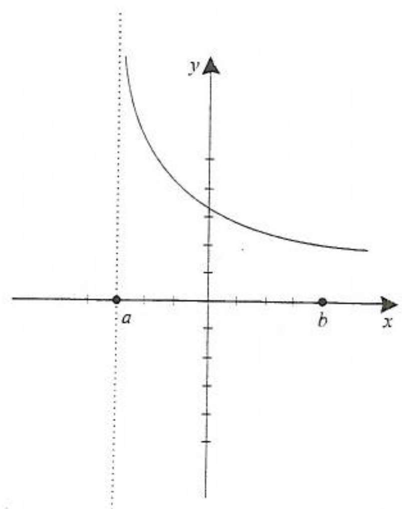

Fig. 5.5

Therefore it is natural to consider the area of the entire region, over the interval $(a, b]$, to be

$$\lim_{\epsilon\to 0^+}\int_{a+\epsilon}^{b} f(x)\,dx.$$

This is just the improper integral

$$\text{Area} = \int_{a}^{b} f(x)\,dx.$$

EXAMPLE 5.20

Calculate the area above the x-axis and under the curve

$$y = \frac{1}{x \cdot \ln^{4/3} x}, \qquad 0 < x \leq 1/2.$$

SOLUTION

According to the preceding discussion, this area is equal to the value of the improper integral

$$\int_0^{1/2} \frac{1}{x \cdot \ln^{4/3} x}\, dx = \lim_{\epsilon \to 0^+} \int_\epsilon^{1/2} \frac{1}{x \cdot \ln^{4/3} x}\, dx.$$

For clarity we let $\varphi(x) = \ln x$, $\varphi'(x) = 1/x$. Then the (indefinite) integral becomes

$$\int \frac{\varphi'(x)}{\varphi^{4/3}(x)}\, dx = -\frac{3}{\varphi^{1/3}(x)}.$$

Thus

$$\lim_{\epsilon \to 0^+} \int_\epsilon^{1/2} \frac{1}{x \cdot \ln^{4/3} x}\, dx = \lim_{\epsilon \to 0^+} - \frac{3}{\ln^{1/3} x}\Bigg|_\epsilon^{1/2}$$

$$= \lim_{\epsilon \to 0^+} \left[\frac{-3}{[-\ln 2]^{1/3}} - \frac{-3}{[\ln \epsilon]^{1/3}}\right].$$

Now as $\epsilon \to 0$ then $\ln \epsilon \to -\infty$ hence $1/[\ln \epsilon]^{1/3} \to 0$. We conclude that our improper integral converges and the area under the curve and above the x-axis equals $3/[\ln 2]^{1/3}$.

5.4 More on Improper Integrals

5.4.1 INTRODUCTION

Suppose that we want to calculate the integral of a continuous function $f(x)$ over an unbounded interval of the form $[A, +\infty)$ or $(-\infty, B]$. The theory of the integral that we learned earlier does not cover this situation, and some new concepts are needed. We treat improper integrals on infinite intervals in this section, and give some applications at the end.

5.4.2 THE INTEGRAL ON AN INFINITE INTERVAL

Let f be a continuous function whose domain contains an interval of the form $[A, +\infty)$. The value of the improper integral

$$\int_A^{+\infty} f(x)\,dx$$

is defined to be

$$\lim_{N \to +\infty} \int_A^N f(x)\,dx.$$

Similarly, we have: Let g be a continuous function whose domain contains an interval of the form $(-\infty, B]$. The value of the improper integral

$$\int_{-\infty}^B g(x)\,dx$$

is defined to be

$$\lim_{M \to -\infty} \int_M^B f(x)\,dx.$$

EXAMPLE 5.21

Calculate the improper integral

$$\int_1^{+\infty} x^{-3}\,dx.$$

SOLUTION

We do this problem by evaluating the limit

$$\begin{aligned}\lim_{N \to +\infty} \int_1^N x^{-3}\,dx &= \lim_{N \to +\infty} \left[-(1/2)x^{-2}\Big|_1^N \right] \\ &= \lim_{N \to +\infty} -(1/2)\left[N^{-2} - 1^{-2}\right] \\ &= \frac{1}{2}.\end{aligned}$$

We conclude that the integral converges and has value $1/2$.

EXAMPLE 5.22

Evaluate the improper integral

$$\int_{-\infty}^{-32} x^{-1/5}\,dx.$$

SOLUTION

We do this problem by evaluating the limit

$$\lim_{M\to-\infty}\int_M^{-32} x^{-1/5}\,dx = \lim_{M\to-\infty} \frac{5}{4}x^{4/5}\Big|_M^{-32}$$
$$= \lim_{M\to-\infty} \frac{5}{4}\left[(-32)^{4/5} - M^{4/5}\right]$$
$$= \lim_{M\to-\infty} \frac{5}{4}\left[16 - M^{4/5}\right].$$

This limit equals $-\infty$. Therefore the integral diverges.

You Try It: Evaluate $\int_1^{\infty}(1+x)^{-3}\,dx$.

Sometimes we have occasion to evaluate a doubly infinite integral. We do so by breaking the integral up into two separate improper integrals, each of which can be evaluated with just one limit.

EXAMPLE 5.23

Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

SOLUTION

The interval of integration is $(-\infty, +\infty)$. To evaluate this integral, we break the interval up into two pieces:

$$(-\infty, +\infty) = (-\infty, 0] \cup [0, +\infty).$$

(The choice of zero as a place to break the interval is not important; any other point would do in this example.) Thus we will evaluate separately the integrals

$$\int_0^{+\infty} \frac{1}{1+x^2}\,dx \quad \text{and} \quad \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx.$$

For the first one we consider the limit

$$\lim_{N\to+\infty}\int_0^{N} \frac{1}{1+x^2}\,dx = \lim_{N\to+\infty} \mathrm{Tan}^{-1}x\Big|_0^N$$
$$= \lim_{N\to+\infty}\left[\mathrm{Tan}^{-1}N - \mathrm{Tan}^{-1}0\right]$$
$$= \frac{\pi}{2}.$$

The second integral is evaluated similarly:

$$\begin{aligned}\lim_{M\to-\infty}\int_M^0 \frac{1}{1+x^2}\,dx &= \lim_{M\to-\infty} \mathrm{Tan}^{-1}x\Big|_M^0 \\ &= \lim_{M\to-\infty}\left[\mathrm{Tan}^{-1}0 - \mathrm{Tan}^{-1}M\right] \\ &= \frac{\pi}{2}.\end{aligned}$$

Since each of the integrals on the half line is convergent, we conclude that the original improper integral over the entire real line is convergent and that its value is

$$\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

You Try It: Discuss $\int_1^\infty (1+x)^{-1}\,dx$.

5.4.3 SOME APPLICATIONS

Now we may use improper integrals over infinite intervals to calculate area.

EXAMPLE 5.24

Calculate the area under the curve $y = 1/[x\cdot(\ln x)^4]$ and above the x-axis, $2 \le x < \infty$.

SOLUTION

The area is given by the improper integral

$$\int_2^{+\infty} \frac{1}{x\cdot(\ln x)^4}\,dx = \lim_{N\to+\infty}\int_2^N \frac{1}{x\cdot(\ln x)^4}\,dx.$$

For clarity, we let $\varphi(x) = \ln x$, $\varphi'(x) = 1/x$. Thus the (indefinite) integral becomes

$$\int \frac{\varphi'(x)}{\varphi^4(x)}\,dx = -\frac{1/3}{\varphi^3(x)}.$$

Thus

$$\begin{aligned}\lim_{N\to+\infty}\int_2^N \frac{1}{x\cdot(\ln x)^4}\,dx &= \lim_{N\to+\infty}\left[-\frac{1/3}{\ln^3 x}\right]_2^N \\ &= \lim_{N\to+\infty} -\left[\frac{1/3}{\ln^3 N}-\frac{1/3}{\ln^3 2}\right] \\ &= \frac{1/3}{\ln^3 2}.\end{aligned}$$

Thus the area under the curve and above the x-axis is $1/(3\ln^3 2)$.

EXAMPLE 5.25

Because of inflation, the value of a dollar decreases as time goes on. Indeed, this decrease in the value of money is directly related to the continuous compounding of interest. For if one dollar today is invested at 6% continuously compounded interest for ten years then that dollar will have grown to $e^{0.06\cdot 10}$ = \$1.82 (see Section 6.5 for more detail on this matter). This means that a dollar in the currency of ten years from now corresponds to only $e^{-0.06\cdot 10}$ = \$0.55 in today's currency.

Now suppose that a trust is established in your name which pays $2t+50$ dollars per year for every year in perpetuity, where t is time measured in years (here the present corresponds to time $t = 0$). Assume a constant interest rate of 6%, and that all interest is re-invested. What is the total value, in today's dollars, of all the money that will ever be earned by your trust account?

SOLUTION

Over a short time increment $[t_{j-1}, t_j]$, the value *in today's currency* of the money earned is about

$$(2t_j + 50) \cdot \left(e^{-0.06\cdot t_j}\right) \cdot \Delta t_j .$$

The corresponding sum over time increments is

$$\sum_j (2t_j + 50) \cdot e^{-0.06\cdot t_j} \Delta t_j .$$

This in turn is a Riemann sum for the integral

$$\int (2t + 50)e^{-0.06t}\, dt .$$

If we want to calculate the value in today's dollars of all the money earned from now on, in perpetuity, this would be the value of the improper integral

$$\int_0^{+\infty} (2t + 50)e^{-0.06t}\, dt .$$

This value is easily calculated to be \$1388.89, rounded to the nearest cent.

You Try It: A trust is established in your name which pays $t + 10$ dollars per year for every year in perpetuity, where t is time measured in years (here the present corresponds to time $t = 0$). Assume a constant interest rate of 4%. What is the total value, in today's dollars, of all the money that will ever be earned by your trust account?

Exercises

1. If possible, use l'Hôpital's Rule to evaluate each of the following limits. In each case, check carefully that the hypotheses of l'Hôpital's Rule apply.

(a) $\lim_{x \to 0} \frac{\cos x - 1}{x^2 - x^3}$

(b) $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2 + x^4}$

(c) $\lim_{x \to 0} \frac{\cos x}{x^2}$

(d) $\lim_{x \to 1} \frac{[\ln x]^2}{(x - 1)}$

(e) $\lim_{x \to 2} \frac{(x-2)^3}{\sin(x-2) - (x-2)}$

(f) $\lim_{x \to 1} \frac{e^x - 1}{x - 1}$

2. If possible, use l'Hôpital's Rule to evaluate each of the following limits. In each case, check carefully that the hypotheses of l'Hôpital's Rule apply.

(a) $\lim_{x \to +\infty} \frac{x^3}{e^x - x^2}$

(b) $\lim_{x \to +\infty} \frac{\ln x}{x}$

(c) $\lim_{x \to +\infty} \frac{e^{-x}}{\ln[x/(x+1)]}$

(d) $\lim_{x \to +\infty} \frac{\sin x}{e^{-x}}$

(e) $\lim_{x \to -\infty} \frac{e^x}{1/x}$

(f) $\lim_{x \to -\infty} \frac{\ln |x|}{e^{-x}}$

3. If possible, use some algebraic manipulations, plus l'Hôpital's Rule, to evaluate each of the following limits. In each case, check carefully that the hypotheses of l'Hôpital's Rule apply.

(a) $\lim_{x \to +\infty} x^3 e^{-x}$

(b) $\lim_{x \to +\infty} x \cdot \sin[1/x]$

(c) $\lim_{x\to+\infty} \ln[x/(x+1)] \cdot (x+1)$

(d) $\lim_{x\to+\infty} \ln x \cdot e^{-x}$

(e) $\lim_{x\to-\infty} e^{2x} \cdot x^2$

(f) $\lim_{x\to 0} x \cdot e^{1/x}$

4. Evaluate each of the following improper integrals. In each case, be sure to write the integral as an appropriate limit.

(a) $\int_0^1 x^{-3/4}\,dx$

(b) $\int_1^3 (x-3)^{-4/3}\,dx$

(c) $\int_{-2}^2 \frac{1}{(x+1)^{1/3}}\,dx$

(d) $\int_{-4}^6 \frac{x}{(x-1)(x+2)}\,dx$

(e) $\int_4^8 \frac{x+4}{(x-8)^{1/3}}\,dx$

(f) $\int_0^3 \frac{\sin x}{x^2}\,dx$

5. Evaluate each of the following improper integrals. In each case, be sure to write the integral as an appropriate limit.

(a) $\int_1^\infty e^{-3x}\,dx$

(b) $\int_2^\infty x^2 e^{-x}\,dx$

(c) $\int_0^\infty x \ln x\,dx$

(d) $\int_1^\infty \frac{dx}{1+x^2}$

(e) $\int_1^\infty \frac{dx}{x}$

(f) $\int_{-\infty}^{-1} \frac{dx}{x^2+x}$

Chapter 5

1. (a) $\lim_{x\to 0}(\cos x - 1) = 0$ and $\lim_{x\to 0} x^2 - x^3 = 0$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2 - x^3} = \lim_{x\to 0} \frac{-\sin x}{2x - 3x^2}.$$

Now l'Hôpital's Rule applies again to yield

$$= \lim_{x\to 0} \frac{-\cos x}{2 - 6x} = -\frac{1}{2}.$$

(b) $\lim_{x\to 0} e^{2x} - 1 - 2x = 0$ and $\lim_{x\to 0} x^2 + x^4 = 0$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x^2 + x^4} = \lim_{x\to 0} \frac{2e^{2x} - 2}{2x + 4x^3}.$$

l'Hôpital's Rule applies again to yield

$$= \lim_{x\to 0} \frac{4e^{2x}}{2 + 12x^2} = 2.$$

(c) $\lim_{x\to 0} \cos x \neq 0$, so l'Hôpital's Rule does not apply. In fact the limit does not exist.

(d) $\lim_{x\to 1} [\ln x]^2 = 0$ and $\lim_{x\to 1}(x - 1) = 0$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to 1} \frac{[\ln x]^2}{(x-1)} = \lim_{x\to 1} \frac{[2\ln x]/x}{1} = 0.$$

(e) $\lim_{x\to 2}(x-2)^3 = 0$ and $\lim_{x\to 2} \sin(x-2) - (x-2) = 0$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to 2} \frac{(x-2)^3}{\sin(x-2) - (x-2)} = \lim_{x\to 2} \frac{3(x-2)^2}{\cos(x-2) - 1}.$$

Now l'Hôpital's Rule applies again to yield

$$= \lim_{x\to 2} \frac{6(x-2)}{-\sin(x-2)}.$$

We apply l'Hôpital's Rule one last time to obtain

$$= \lim_{x\to 2} \frac{6}{-\cos(x-2)} = -6.$$

(f) $\lim_{x\to 1}(e^x - 1) = 0$ and $\lim_{x\to 1}(x - 1) = 0$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to 1} \frac{e^x - 1}{x - 1} = \lim_{x\to 1} \frac{e^x}{1} = e.$$

2. (a) $\lim_{x\to +\infty} x^3 = \lim_{x\to +\infty}(e^x - x^2) = +\infty$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to +\infty} \frac{x^3}{e^x - x^2} = \lim_{x\to +\infty} \frac{3x^2}{e^x - 2x}.$$

l'Hôpital's Rule applies again to yield

$$= \lim_{x\to+\infty} \frac{6x}{e^x - 2}.$$

l'Hôpital's Rule applies one more time to finally yield

$$\lim_{x\to+\infty} \frac{6}{e^x} = 0.$$

(b) $\lim_{x\to+\infty} \ln x = \lim_{x\to+\infty} x = +\infty$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1/x}{1} = 0.$$

(c) $\lim_{x\to+\infty} e^{-x} = \lim_{x\to+\infty} \ln[x/(x+1)] = 0$ so l'Hôpital's Rule applies. Thus

$$\lim_{x\to+\infty} \frac{e^{-x}}{\ln[x/(x+1)]} = \lim_{x\to+\infty} \frac{-e^{-x}}{1/x - 1/[x+1]}.$$

It is convenient to rewrite this expression as

$$\lim_{x\to+\infty} \frac{x^2 + x}{-e^x}.$$

Now l'Hôpital's Rule applies once more to yield

$$\lim_{x\to+\infty} \frac{2x+1}{-e^x}.$$

We apply l'Hôpital's Rule a last time to obtain

$$= \lim_{x\to+\infty} \frac{2}{-e^x} = 0.$$

(d) Since $\lim_{x\to+\infty} \sin x$ does not exist, l'Hôpital's Rule does not apply. In fact the requested limit does not exist.

(e) It is convenient to rewrite this limit as

$$\lim_{x\to-\infty} \frac{x}{e^{-x}}.$$

Since $\lim_{x\to-\infty} x = \lim_{x\to-\infty} e^{-x} = \pm\infty$, l'Hôpital's Rule applies. Thus

$$\lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0.$$

(f) Since $\lim_{x\to-\infty} \ln|x| = \lim_{x\to-\infty} e^{-x} = +\infty$, l'Hôpital's Rule applies. Thus

$$\lim_{x\to-\infty} \frac{\ln|x|}{e^{-x}} = \lim_{x\to-\infty} \frac{1/x}{-e^{-x}} = 0.$$

3. (a) We write the limit as $\lim_{x\to+\infty} \frac{x^3}{e^x}$. Since $\lim_{x\to+\infty} x^3 = \lim_{x\to+\infty} e^x = +\infty$, l'Hôpital's Rule applies. Thus

$$\lim_{x\to+\infty} x^3 e^{-x} = \lim_{x\to+\infty} \frac{x^3}{e^x} = \lim_{x\to+\infty} \frac{3x^2}{e^x}.$$

We apply l'Hôpital's Rule again to obtain

$$= \lim_{x\to+\infty} \frac{6x}{e^x}.$$

Applying l'Hôpital's Rule one last time yields

$$= \lim_{x\to+\infty} \frac{6}{e^x} = 0.$$

(b) We write the limit as $\lim_{x\to+\infty} \frac{\sin(1/x)}{1/x}$. Since $\lim_{x\to+\infty} \sin(1/x) = \lim_{x\to+\infty} 1/x = 0$, l'Hôpital's Rule applies. Hence

$$\begin{aligned}\lim_{x\to+\infty} x \cdot \sin[1/x] &= \lim_{x\to+\infty} \frac{\sin(1/x)}{1/x}\\ &= \lim_{x\to+\infty} \frac{[\cos(1/x)]\cdot[-1/x^2]}{-1/x^2}\\ &= \lim_{x\to+\infty} \frac{\cos(1/x)}{1} = 1.\end{aligned}$$

(c) We rewrite the limit as $\lim_{x\to+\infty} \frac{\ln[x/(x+1)]}{1/(x+1)}$. Since $\lim_{x\to+\infty} \ln[x/(x+1)] = \lim_{x\to+\infty} 1/(x+1) = 0$, l'Hôpital's Rule applies. Thus

$$\begin{aligned}\lim_{x\to+\infty} \ln[x/(x+1)]\cdot(x+1) &= \lim_{x\to+\infty} \frac{\ln[x/(x+1)]}{1/(x+1)}\\ &= \lim_{x\to+\infty} \frac{[(x+1)/x]\cdot[1/(x+1)^2]}{-1/(x+1)^2}\\ &= \lim_{x\to+\infty} \frac{-(x+1)}{x}.\end{aligned}$$

Now l'Hôpital's Rule applies again and we obtain

$$= \lim_{x\to+\infty} \frac{-1}{1} = -1.$$

(d) We rewrite the limit as $\lim_{x\to+\infty} \frac{[\ln x]}{e^x}$. Since $\lim_{x\to+\infty} \ln x = \lim_{x\to+\infty} e^x = +\infty$, l'Hôpital's Rule applies. Thus

$$\lim_{x\to+\infty} \ln x \cdot e^{-x} = \lim_{x\to+\infty} \frac{\ln x}{e^x} = \lim_{x\to+\infty} \frac{1/x}{e^x} = 0.$$

(e) We write the limit as $\lim_{x\to-\infty} \frac{x^2}{e^{-2x}}$. Since $\lim_{x\to-\infty} \lim x^2 = \lim_{x\to-\infty} e^{-2x} = 0$, l'Hôpital's Rule applies. Thus

$$\lim_{x\to-\infty} e^{2x} \cdot x^2 = \lim_{x\to-\infty} \frac{x^2}{e^{-2x}} = \lim_{x\to-\infty} \frac{2x}{-2e^{-2x}}.$$

l'Hôpital's Rule applies one more time to yield

$$= \lim_{x\to-\infty} \frac{2}{4e^{-2x}} = 0.$$

(f) We rewrite the limit as $\lim_{x\to0} \frac{e^{1/x}}{[1/x]}$. Since $\lim_{x\to0} e^{1/x} = \lim_{x\to0} 1/x = +\infty$, l'Hôpital's Rule applies. Thus

$$\lim_{x\to0} x \cdot e^{1/x} = \lim_{x\to0} \frac{e^{1/x}}{1/x} = \lim_{x\to0} \frac{e^{1/x}\cdot[-1/x^2]}{-1/x^2} = \lim_{x\to0} \frac{e^{1/x}}{1} = +\infty.$$

4. We do (a), (b), (c), (d).

(a) $$\int_0^1 x^{-3/4}\,dx = \lim_{\epsilon\to0^+} \int_\epsilon^1 x^{-3/4}\,dx = \lim_{\epsilon\to0^+} \left[\frac{x^{1/4}}{1/4}\right]_\epsilon^1$$

$$= \lim_{\epsilon\to0^+} \left(\frac{1^{1/4}}{1/4} - \frac{\epsilon^{1/4}}{1/4}\right) = 4.$$

(b) $$\int_1^3 (x-3)^{-4/3}\,dx = \lim_{\epsilon\to0^+} \int_1^{3-\epsilon} (x-3)^{-4/3}\,dx$$

$$= \lim_{\epsilon\to0^+} \left[\frac{(x-3)^{-1/3}}{-1/3}\right]_1^{3-\epsilon} = \lim_{\epsilon\to0^+} \left(\frac{-\epsilon^{-1/3}}{-1/3} - \frac{-2^{-1/3}}{-1/3}\right).$$ But the limit does not exist; so the integral does not converge.

(c) $$\int_{-2}^2 \frac{1}{(x+1)^{1/3}}\,dx = \lim_{\epsilon\to0^+} \int_{-2}^{-1-\epsilon} \frac{1}{(x+1)^{1/3}}\,dx$$

$$+ \lim_{\epsilon \to 0^+} \int_{-1+\epsilon}^{2} \frac{1}{(x+1)^{1/3}}\, dx$$

$$= \lim_{\epsilon \to 0^+} \left[\frac{(x+1)^{2/3}}{2/3} \right]_{-2}^{-1-\epsilon} + \lim_{\epsilon \to 0^+} \left[\frac{(x+1)^{2/3}}{2/3} \right]_{-1+\epsilon}^{2}$$

$$= \lim_{\epsilon \to 0^+} \left(\frac{(-\epsilon)^{2/3}}{2/3} - \frac{(-1)^{2/3}}{2/3} \right) + \lim_{\epsilon \to 0^+} \left(\frac{3^{2/3}}{2/3} - \frac{(\epsilon)^{2/3}}{2/3} \right)$$

$$= \frac{3}{2} \cdot \left(3^{2/3} - 1 \right).$$

(d) $$\int_{-4}^{6} \frac{x}{(x-1)(x+2)}\, dx = \lim_{\epsilon \to 0^+} \int_{-4}^{-2-\epsilon} \frac{x}{(x-1)(x+2)}\, dx$$

$$+ \lim_{\epsilon \to 0^+} \int_{-2+\epsilon}^{0} \frac{x}{(x-1)(x+2)}\, dx + \lim_{\epsilon \to 0^+} \int_{0}^{1-\epsilon} \frac{x}{(x-1)(x+2)}\, dx$$

$$+ \lim_{\epsilon \to 0^+} \int_{1+\epsilon}^{6} \frac{x}{(x-1)(x+2)}\, dx.$$ Now

$$\frac{x}{(x-1)(x+2)} = \frac{1/3}{x-1} + \frac{2/3}{x+2}.$$

Therefore

$$\int_{-4}^{6} \frac{x}{(x-1)(x+2)}\, dx$$

$$= \lim_{\epsilon \to 0^+} \int_{-4}^{-2-\epsilon} \frac{1/3}{x-1} + \frac{2/3}{x+2}\, dx + \lim_{\epsilon \to 0^+} \int_{-2+\epsilon}^{0} \frac{1/3}{x-1} + \frac{2/3}{x+2}\, dx$$

$$+ \lim_{\epsilon \to 0^+} \int_{0}^{1-\epsilon} \frac{1/3}{x-1} + \frac{2/3}{x+2}\, dx + \lim_{\epsilon \to 0^+} \int_{1+\epsilon}^{6} \frac{1/3}{x-1} + \frac{2/3}{x+2}\, dx$$

$$= \lim_{\epsilon \to 0^+} \left[\frac{1}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| \right]_{-4}^{-2-\epsilon}$$

$$+ \lim_{\epsilon \to 0^+} \left[\frac{1}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| \right]_{-2+\epsilon}^{0}$$

$$+ \lim_{\epsilon \to 0^+} \left[\frac{1}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| \right]_{0}^{1-\epsilon}$$

$$+ \lim_{\epsilon \to 0^+} \left[\frac{1}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| \right]_{1+\epsilon}^{6}.$$

Now this equals

$$\lim_{\epsilon\to 0^+}\left(\frac{1}{3}\cdot\ln|-3-\epsilon|+\frac{2}{3}\ln\epsilon\right)-\left(\frac{1}{3}\cdot\ln 5+\frac{2}{3}\ln 2\right)+\text{etc.}$$

The second limit does not exist, so the original integral does not converge.

5. We do (a), (b), (c), (d).

(a) $$\int_1^\infty e^{-3x}\,dx=\lim_{N\to+\infty}e^{-3x}\,dx=\lim_{N\to+\infty}\left[\frac{e^{-3x}}{-3}\right]_1^N$$
$$=\lim_{N\to+\infty}\left(\frac{e^{-3N}}{-3}-\frac{e^{-3}}{-3}\right)=\frac{e^{-3}}{3}.$$

(b) $$\int_2^\infty x^2e^{-x}\,dx=\lim_{N\to+\infty}\int_2^N x^2e^{-x}\,dx$$
$$=\lim_{N\to+\infty}\left[-e^{-x}x^2-2xe^{-x}-2e^{-x}\right]_2^N$$
$$=\lim_{N\to+\infty}\left[\left(-e^{-N}N^2-2Ne^{-N}-2e^{-N}\right)\right.$$
$$\left.-\left(-e^{-2}2^2-2\cdot 2\cdot e^{-2}-2e^{-2}\right)\right]$$
$$=e^{-2}2^2+4e^{-2}+2e^{-2}.$$

(c) $$\int_0^\infty x\ln x\,dx=\lim_{\epsilon\to 0^+}\int_\epsilon^1 x\ln x\,dx+\lim_{N\to+\infty}\int_1^N x\ln x\,dx$$
$$=\lim_{\epsilon\to^+}\left[x\ln x-x\right]_\epsilon^1+\lim_{N\to+\infty}\left[x\ln x-x\right]_1^N$$
$$=\lim_{\epsilon\to 0^+}\left[(1\cdot\ln 1-1)-(\epsilon\cdot\ln\epsilon-\epsilon)\right]$$
$$+\lim_{N\to+\infty}\left[(N\cdot\ln N-N)-(1\ln 1-1)\right]$$
$$=\lim_{\epsilon\to 0^+}\left[-1+\epsilon\right]+\lim_{N\to+\infty}\left[N\ln N-N+1\right]$$
$$=\lim_{N\to+\infty}\left[N\ln N-N\right].$$ This last limit diverges, so the integral diverges.

(d) $$\int_1^\infty\frac{dx}{1+x^2}=\lim_{N\to+\infty}\int_1^N\frac{dx}{1+x^2}=\lim_{N\to+\infty}\left[\arctan x\right]_1^N$$
$$=\lim_{N\to+\infty}\left(\arctan N-\arctan 1\right)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$