HOW TO ACE the rest of Calculus

THE STREETWISE GUIDE

Including MultiVariable Calculus

"What a great book! It's short, it's funny, and it reveals the secrets of the calculus guild. What more could you want?"

Fernando Gouvêa
Editor, MAA Online

COLIN ADAMS
ABIGAIL THOMPSON
JOEL HASS

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{10^n}$. Taking absolute values gives a geometric series that converges, so this series does also. Or use the alternating series test.

3. $\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}$. The factorial tells us to use the ratio test. It converges.

4. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$. The basic comparison test doesn't work. Use the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ to see that it diverges.

5. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$. Use the basic comparison test with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ to see that the series diverges.

6. $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$. This definitely calls for the ratio test (look at all those factorials), and the series converges.

Hey, whatever pops your weasel.

4.12 Taylor series

Polynomials are the type of functions we can get to know intimately. They're open and frank and don't hold back information. Take $f(x) = 2 - 3x + 7x^2$, for instance. This polynomial will tell us anything we want to know and more. We say, "Hey there, f, what's your value at 4?" and $f(x)$ responds, "Well, just take $2 - 3(4) + 7(4^2) = 2 - 12 + 112 = 102$ and there you go. And by the way, I have a weird rash that itches like crazy, but it's getting better."

That's just the kind of functions they are. The amazing part is that just with some additions and multiplications, we can get the value of this polynomial anywhere we want.

The same works for a general polynomial of the form:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

It can be evaluated at any point using just addition and multiplication, operations we've been familiar with since way before puberty. No secrets, no subterfuge. What you see is what you get.

The same holds true for another form of polynomial:

$$g(x) = b_0 + b_1(x - a) + b_2(x - a)^2 + \cdots + b_n(x - a)^n$$

This is a polynomial in a different pantsuit. If we multiplied out each of the terms $(x-a)^2, (x-a)^3, \ldots, (x-a)^n$, and then cleaned up, we would have a polynomial that looked just like the first expression for $f(x)$.

But there are other less friendly functions. Functions that prefer to stay in the basement where no one will try to start up a conversation. Yes, we're talking about $\sin x$, $\ln x$, e^x and the like. And let's not mention $\arctan x$.

Try asking $\ln x$, "So, umm, what's your value at 3?" You'll get a response like, "Why do you want to know? What's it to you? Why're you prying into my affairs? You some kind of hot shot mathhead? Think you can just walk up here and with a little addition and multiplication, you have me figured out? Well, you're out of luck, base 2 brain. It's none of your exponential business what my value at 3 is. Now clear out of here, before I kick you in your tangenital component."

You get the idea, unfriendly. And in particular, not about to share with you its value at 3. So what do you do?

Well, that's what this section is all about. We'll replace those nasty functions with our dear friends the polynomials. The hard-to-compute functions will be approximated by the easy-to-compute polynomials. Pretty clever, huh? Let's start with an extended example.

Example 1 Approximate $f(x) = e^x$ by a polynomial for x near 0.

If we use a fancier high-degree polynomial, the approximation will get better. Let's start simply.

Phase 1 Approximate $f(x) = e^x$ by a polynomial of the form

$$y = a_0 + a_1 x$$

This is a so-called linear polynomial, meaning that its graph is a line. Suppose we want to approximate e^x by a line for values of x near 0. The best choice for such a line is the tangent line to e^x at $(x, y) = (0, 1)$, as in Figure 4.7.

Notice that the tangent line has the same value and the same first derivative as e^x does at $x = 0$. The slope of the tangent line is the derivative of e^x at $x = 0$, since this follows from the definition of the derivative. But $d(e^x)/dx = e^x$, so

$$\left.\frac{d(e^x)}{dx}\right|_{x=0} = e^0 = 1$$

So the slope of the line is 1. Since the line has y-intercept 1, the equation of the line is

$$y = x + 1$$

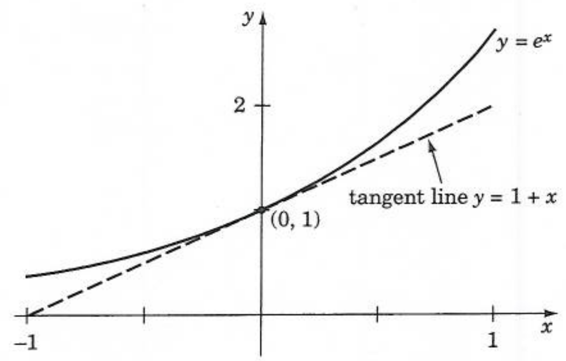

Figure 4.7 Approximating $y = e^x$ by a line.

We have found a line that approximates $y = e^x$ for x near 0, namely, $y = 1 + x$. So

$$e^x \approx 1 + x$$

How accurate is it? Let's evaluate both e^x and $1 + x$ at a small value of x, say $x = 0.1$. Then

$$e^{0.1} \approx 1.105170918$$

On the other hand, $1 + x$ evaluated at $x = 0.1$ is 1.1. So we can see that this "linear approximation" is valid to one decimal place. Not too shabby.

Phase 2 Now we will approximate $f(x) = e^x$ by a second-degree polynomial of the form $g(x) = a_0 + a_1x + a_2x^2$ for values of x near 0. Instead of approximating the graph of e^x by a line, we will be approximating it by a parabola. This should allow us to stay closer to the graph for a longer period of time. This round, we will find the values of a_0, a_1, and a_2 that will ensure that $f(x)$ and the polynomial $g(x)$ have the same value, the same first derivative, and the same second derivative, all at $x = 0$. Then the two functions should be pretty similar, at least for values of x close to $x = 0$.

$$\begin{aligned}
f(x) &= e^x & &\text{so } f(0) = e^0 = 1 \\
f'(x) &= e^x & &\text{so } f'(0) = e^0 = 1 \\
f''(x) &= e^x & &\text{so } f''(0) = e^0 = 1 \\
g(x) &= a_0 + a_1x + a_2x^2 & &\text{so } g(0) = a_0 \\
g'(x) &= a_1 + 2a_2x & &\text{so } g'(0) = a_1 \\
g''(x) &= 2a_2 & &\text{so } g''(0) = 2a_2
\end{aligned}$$

For the values and derivatives of these two functions to match at $x = 0$, we need:

$$\begin{aligned} 1 &= f(0) = g(0) = a_0 & \text{so } a_0 &= 1 \\ 1 &= f'(0) = g'(0) = a_1 & \text{so } a_1 &= 1 \\ 1 &= f''(0) = g''(0) = 2a_2 & \text{so } a_2 &= 1/2 \end{aligned}$$

therefore, we get that $g(x) = 1 + x + x^2/2$, and so

$$e^x \approx 1 + x + \frac{x^2}{2} \qquad \text{for } x \text{ near } 0$$

This is the same as the first linear approximation that we found, only now we have an additional term involving x^2. The parabolic curve giving us our approximation appears in Figure 4.8.

How good is this approximation? Let's try it when $x = 0.1$. Remember, $e^{0.1} \approx 1.105170918$. Now $1 + x + x^2/2$ at $x = 0.1$ gives 1.105. Not a bad approximation, wouldn't you say? We have picked up another decimal place worth of accuracy.

Phase 3 Let's go one more step and find a third-degree polynomial $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ to approximate $f(x) = e^x$ for x near 0.

Again, $f(0) = e^0 = 1, f'(0) = e^0 = 1, f''(0) = e^0 = 1$, and $f'''(0) = e^0 = 1$.

$$\begin{aligned} g(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 & \text{so } g(0) &= a_0 \\ g'(x) &= a_1 + 2a_2x + 3a_3x^2 & \text{so } g'(0) &= a_1 \\ g''(x) &= 2a_2 + 3(2)a_3x & \text{so } g''(0) &= 2a_2 \\ g'''(x) &= 3(2)a_3 & \text{so } g'''(0) &= 6a_3 \end{aligned}$$

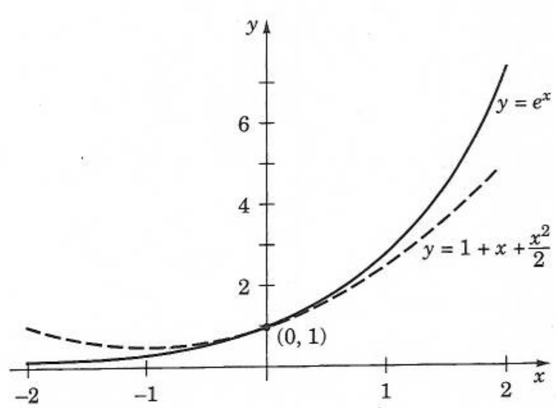

Figure 4.8 Approximating $y = e^x$ by a parabola.

For the values and derivatives of these two functions to match at $x = 0$, we need:

$$\begin{aligned}
1 &= f(0) = g(0) = a_0 && \text{so } a_0 = 1 \\
1 &= f'(0) = g'(0) = a_1 && \text{so } a_1 = 1 \\
1 &= f''(0) = g''(0) = 2a_2 && \text{so } a_2 = 1/2 \\
1 &= f'''(0) = g'''(0) = 3(2)a_3 && \text{so } a_3 = 1/6
\end{aligned}$$

This gives

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

which is the same as our previous approximation but with an extra $x^3/6$ tacked on the end. The cubic curve that is our new approximation to e^x for x near 0 appears in Figure 4.9.

How accurate is this improved approximation? Remember that $e^{0.1} \approx 1.105170918$. We calculate that $1 + x + x^2/2 + x^3/6$ is $1.1051666\ldots$ when $x = 0.1$. Wow! That's a good approximation. Makes you want to call CNN.

We can see a pattern developing. If we continue to add additional terms in this manner, we would find:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

This looks a lot like a power series!

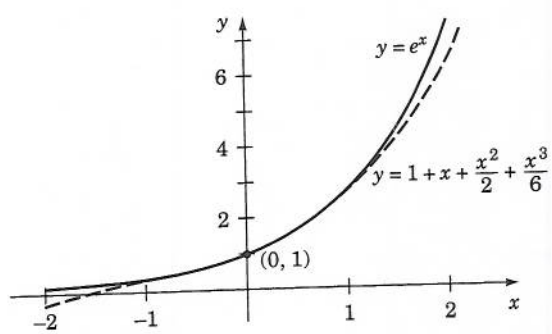

Figure 4.9 Approximating $y = e^x$ by a cubic curve.

Now, you get the idea! We take some ugly difficult function, call it $f(x)$, and we want to find good approximations for $f(x)$ for values of x near some fixed value a. (In our example above, we used $a = 0$.)

So we find a polynomial that behaves very much like $f(x)$ for values of x near a. In general, we will use a polynomial of the form

$$g(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots + b_n(x-a)^n$$

To make the polynomial behave like $f(x)$ for x near a, we need its value at $x = a$ and the values of its first n-derivatives at $x = a$ to match the same values for $f(x)$.

That is to say, we ask that

$$\begin{aligned} f(a) &= g(a) \\ f'(a) &= g'(a) \\ f''(a) &= g''(a) \\ f'''(a) &= g'''(a) \\ &\cdots\cdots\cdots \\ f^{(n)}(a) &= g^{(n)}(a) \end{aligned}$$

Now, straightforward differentiation gives

$$\begin{aligned} g(x) &= b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots + b_n(x-a)^n \\ g'(x) &= b_1 + 2b_2(x-a) + 3b_3(x-a)^2 + \cdots + nb_n(x-a)^{n-1} \\ g''(x) &= 2b_2 + 3(2)b_3(x-a) + \cdots + n(n-1)b_n(x-a)^{n-2} \\ g'''(x) &= 3(2)b_3 + \cdots + n(n-1)(n-2)b_n(x-a)^{n-3} \\ &\cdots\cdots\cdots \\ g^{(n)}(x) &= n(n-1)(n-2)\cdots(3)(2)b_n \end{aligned}$$

we must have:

$$\begin{aligned} f(a) &= g(a) = b_0 && \text{so } b_0 = f(a) \\ f'(a) &= g'(a) = b_1 && \text{so } b_1 = f'(a) \\ f''(a) &= g''(a) = 2b_2 && \text{so } b_2 = \frac{f''(a)}{2} \\ f'''(a) &= g'''(a) = 3!b_3 && \text{so } b_3 = \frac{f'''(a)}{3!} \\ &\cdots\cdots\cdots \\ f^{(n)}(a) &= g^{(n)}(a) = n!b_n && \text{so } b_n = \frac{f^{(n)}(a)}{n!} \end{aligned}$$

Substituting these expressions in for the b's gives us one of the most famous and useful formulas in all of mathematics.

Taylor's Approximation

$$f(x) \approx f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

This approximation works well for values of x near a. Often, a is just equal to 0. Then the formula simplifies and sometimes gets a new name.

Maclaurin's Approximation

$$f(x) \approx f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

Let's try an example.

Example 2 Find a third-degree Taylor approximation to $f(x) = \sin x$ for $a = 0$, and use it to approximate $\sin(0.05)$.

Solution Well, $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, and $f'''(x) = -\cos x$.

So $f(0) = \sin 0 = 0$, $f'(0) = \cos 0 = 1$, $f''(0) = -\sin 0 = 0$, and $f'''(0) = -\cos 0 = -1$.

Then

$$\begin{aligned}\sin x &\approx f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3\\ &\approx 0+1x+0+\frac{-1}{3!}x^3\\ &\approx x-\frac{x^3}{6}\end{aligned}$$

Is this a good approximation at $x = 0.05$?

At $x = 0.05$:
$$x-\frac{x^3}{6} = 0.049875$$
$$\sin(0.05) = 0.049979169\ldots$$

Another impressive approximation! Just imagine how good it would have been if we had included a few more terms. In fact, why not do that? It's not hard to see that if we continued to add terms in this example, we would find:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

This is a power series. It is called the *Taylor series* of $\sin x$. When it converges, we can use it to get better and better approximations of our nasty function. But when our power series diverges, it's completely useless.

Hey, whatever tickles your troglodyte.

4.13 Taylor's formula with remainder

Now we get to the end, the remnant, the stuff that's left over when everything else has been snapped up. Yes, it's time for the remainder sale. Time to move out those Taylor series remainders.

Although Taylor's formula gives us a good approximation to $f(x)$ near $x = a$, it's not exact, and there is an error. And sometimes when we use Taylor's formula we absolutely positively have to know how large the error can be. Otherwise, the bridge we designed almost holds cars. Or the new male contraceptive pill we have created almost prevents pregnancy. Fortunately, there is a simple method for bounding the error.

If we take a degree n Taylor polynomial to approximate $f(x)$ near $x = a$, we get:

Taylor's Formula with Remainder

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{Taylor polynomial } P_n(x)}$$
$$+ \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{Remainder } R_n(x)}$$

where c is some number between a and x.

The quantity $R_n(x)$ is the *remainder,* or *error term.* It's the difference between the degree n Taylor polynomial at x, which is just an approximation, and the exact value $f(x)$. Notice that it depends on some unidentified number c that lies somewhere between a and x.

Unfortunately, when we use Taylor approximations, and we want to know how big the possible error is, we don't know which c gives the precise error. So the procedure is to bound the value of $R_n(x)$ for any value of c between a and x and therefore get a bound on how bad the error could possibly be.

Example 1 [Estimating sin (1)] Ever wondered about the value of sin (1) where 1 is in radians? (Not to be confused with first sin, which involved eating an apple in a garden.) It's the kind of thing that can keep you up at night. Sure you could plug into your calculator to see what it is, but that's not very satisfying. We would like to be able to see how the calculator actually finds such a value. So let's answer this question so you'll be well rested in the morning. Estimate sin (1) with a third-degree Taylor polynomial and determine a bound for the accuracy of your estimate.

Solution The third-degree Taylor approximation for $\sin x$ with $a = 0$ is $\sin x \approx x - x^3/3!$. This gives an estimate for the value of sin (1) which is

$$\begin{aligned} \sin(1) &\approx 1 - \frac{1^3}{3!} \\ &\approx 1 - 0.1666\ldots \\ &= 0.8333\ldots \end{aligned}$$

But how accurate is this? The remainder formula tells us that the error $R_3(1)$ is given by

$$R_3(1) = \frac{(\sin x)^{(4)}(c)}{(4)!}(1)^4$$

for some c with $0 \leq c \leq 1$. We don't know what this c is, but amazingly enough, we don't need to know what it is in order to bound $R_3(1)$. Since the fourth derivative $(\sin x)^{(4)}$ is equal to $\sin x$, and the values of $\sin x$ are always between -1 and 1, we know that $|(\sin)^{(4)}(c)| \leq 1$. So

$$\begin{aligned} |R_3(1)| &\leq \frac{1}{4!} = \frac{1}{24} \\ &\leq 0.042 \end{aligned}$$

The error is no larger than this number, no matter what the value of c might be. The remainder formula tells us that the true value of sin (1) lies between $(0.8333\ldots - 0.042)$ and $(0.8333\ldots + 0.042)$. Just to verify this, note that the actual value of sin (1) is $0.8414709848079\ldots$, which is within this range.

Example 2 Estimate $\sqrt{e}$ with a second-degree Taylor series and determine a bound for the accuracy of your estimate. Repeat with a tenth-degree Taylor polynomial.

Solution The second-degree Taylor series approximation for e^x about $a = 0$ is $e^x \approx 1 + x + x^2/2!$. Plugging in $x = 0.5$ gives

$$\sqrt{e} = e^{0.5} \approx 1 + (0.5) + \frac{(0.5)^2}{2!} \approx 1 + 0.5 + 0.125 = 1.625$$

So how accurate is this? Well, the remainder formula tells us that the error is

$$R_2(0.5) = \frac{e^c}{3!}(0.5)^3$$

for some c with $0 \le c \le 0.5$. How big can this be? Well, unlike in the previous example, we don't know that e^c is always less than 1. However, e^x is an increasing function, and $e^c \le e^{0.5}$. Unfortunately, we don't know how big the number $e^{0.5}$ is. In fact that's what we want to find!

But we do know that $e^{0.5} < 4^{0.5} = 2$. So the error $R_2(0.5)$ satisfies

$$|R_2(0.5)| \le \frac{2}{3!}(0.5)^3 = \frac{1}{24} < 0.042$$

The error is less than 0.042, so we know for sure that $e^{0.5}$ is between $1.625 - 0.042$ and $1.625 + 0.042$. That's not so accurate, but hey, we didn't use that many terms.

If we use the tenth-degree polynomial, $e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{10}}{10!}$, then plugging in $x = 0.5$ gives us an estimate for $e^{0.5}$ of 1.648721271. The remainder is now bounded by $\frac{2}{11!}(0.5)^{11}$, which is less than 2.45×10^{-11}, a mighty small error for the approximation. So we know just by doing a few multiplications and additions that the true value of $e^{0.5}$ is 1.648721271, accurate to that many decimal places. Hey, we just did as well as our calculator could do. That means we are as smart as a calculator. (Calculators fall somewhere between toasters and coffeemakers on the appliance intelligence scale.)

And speaking of remainders, did you hear about the two student entrepeneurs from the Math Club who bet their year's tuition that red T-shirts covered with yellow calculus equations would be the big fashion hit on campus that spring? They put all the the funds their parents had sent together with all they could borrow to purchase a mountain of T-shirts. Unfortunately, the end of winter classes came without any orders. Despite a massive ad campaign in the college paper, the phone failed to ring even once. Unsold T-shirts (remainders) filled their dorm room. Without money for tuition, ruin was imminent. Then, a week before fees were due, the phone rang. "This is Bud from the college bookstore" says the voice at the other end. "Hey, word has it that you have red T-shirts with some yellow equations on them. We could use them for our spring promotion. We can pay top dollar." Bud mentioned a massive sum of money, but with one caveat. "I just have to clear it with purchasing because it's such a big amount. If you don't hear from me by 5 p.m. Friday, it's a done deal."

The two students couldn't believe their luck. They sat around nervously as the clock moved toward 5 p.m. Friday, just praying they wouldn't get a call. At 4:55 p.m. the phone rang. Nervously, her heart beating furiously, one of the students picked up the phone. Then, slowly, as she listened, a smile crossed her face. "Good news," she said. "Your car is on fire."

4.14 Some famous Taylor series

We have already mentioned a few of the most famous Taylor series. For instance,

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

and

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}$$

In fact, we can manipulate Taylor series just like functions in order to generate other series. For instance, since $\cos x$ can be obtained by differentiating $\sin x$, a Taylor series for $\cos x$ can be obtained by differentiating the Taylor series for $\sin x$ term by term. The result is

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

What other basic Taylor series do we have? Let's go back to geometric series for just a second. Remember that when $|r| < 1$, we knew that the geometric series

$$a + ar + ar^2 + \cdots$$

converged to $\dfrac{a}{1-r}$. Letting $a = 1$ and replacing r with the variable x, we have

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad \text{for any } |x| < 1}$$

So this is a series representation of $\dfrac{1}{1-x}$. In fact, it agrees with what we would obtain by applying Taylor's formula. Notice though that it is only valid for $|x| < 1$.

Warning The Taylor series of a function may diverge for some or all values of x.

The four basic series above are worth memorizing. They can often be used to derive other series when the need arises.

Example Find a series that gives $\arctan x$ for $|x| < 1$.

Solution You may recall from when we studied antiderivatives that

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

Or you may not. But anyway this was one of the formulas we worked out then.

Now, replacing each x in $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$ by $-t^2$, we obtain

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots$$

Integrating both sides gives us

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x 1 - t^2 + t^4 - \cdots\,dt$$

$$= t\Big|_0^x - \frac{t^3}{3}\Big|_0^x + \frac{t^5}{5}\Big|_0^x - \cdots = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

which is the Taylor series for $\arctan x$ for $|x| < 1$. But this series is only valid when $|x| < 1$ even though $\arctan x$ is defined for all real numbers.

Hey, whatever buttons your cardigan.