Operations on Power Series

Let's start with 2 power series about x_0:

$$f(x) \;=\; \sum_{n=0}^{\infty} a_n(x-x_0)^n \;=\; a_0 \;+\; a_1(x-x_0)^1 \;+\; a_2(x-x_0)^2 \;+\; a_3(x-x_0)^3 \;+\; \dots$$

$$g(x) \;=\; \sum_{n=0}^{\infty} b_n(x-x_0)^n \;=\; b_0 \;+\; b_1(x-x_0)^1 \;+\; b_2(x-x_0)^2 \;+\; b_3(x-x_0)^3 \;+\; \dots$$

each of which converge **absolutely** for $x \in (x_0 - R\ ,\ x_0 + R)$. Let:

$$c \in \mathbb{R} \overset{\text{def}}{=} (-\infty, +\infty) \quad , \quad m \in \mathbb{N} \overset{\text{def}}{=} \{1, 2, 3, \dots\}$$

$$x \in (x_0 - R\ ,\ x_0 + R) \quad , \quad \alpha \in (x_0 - R\ ,\ x_0 + R) \quad , \quad \beta \in (x_0 - R\ ,\ x_0 + R)$$

Then (note we excluded the endpoints of $(x_0 - R\ ,\ x_0 + R)$, ie. we excluded $x = x_0 \pm R$ since things sometimes don't hold at the endpoints):

$$f(x) + g(x) \overset{(*)}{=} \sum_{n=0}^{\infty} (a_n + b_n)\,(x-x_0)^n$$

$$f(x) - g(x) \overset{(*)}{=} \sum_{n=0}^{\infty} (a_n - b_n)\,(x-x_0)^n$$

$$c\ f(x) \overset{(*)}{=} \sum_{n=0}^{\infty} c\, a_n\,(x-x_0)^n$$

$$(x-x_0)^m\ f(x) \overset{(*)}{=} \sum_{n=0}^{\infty} a_n\,(x-x_0)^m\,(x-x_0)^n \;=\; \sum_{n=0}^{\infty} a_n\,(x-x_0)^{m+n}$$

$$D_x\left[f(x)\right] \;=\; D_x\left[\sum_{n=0}^{\infty} a_n(x-x_0)^n\right] \overset{(*)}{=} \sum_{n=0}^{\infty} D_x(a_n(x-x_0)^n) \;=\; \sum_{n=0}^{\infty} n a_n(x-x_0)^{n-1}$$

$$So \quad D_x\left[f(x)\right] \;=\; \sum_{n=1}^{\infty} n a_n(x-x_0)^{n-1}$$

$$\int_\alpha^\beta f(t)\,dt \;=\; \int_\alpha^\beta \left(\sum_{n=0}^{\infty} a_n(t-x_0)^n\right) dt \overset{(*)}{=} \sum_{n=0}^{\infty} \int_\alpha^\beta (a_n(t-x_0)^n)\ dt \;=\; \sum_{n=0}^{\infty} \frac{a_n\,(t-x_0)^{n+1}}{n+1}\Bigg|_{t=\alpha}^{t=\beta}$$

$$So \quad \int_\alpha^\beta f(t)\,dt \;=\; \sum_{n=0}^{\infty} \frac{a_n}{n+1}\,\left[(\beta-x_0)^{n+1} \;-\; (\alpha-x_0)^{n+1}\right]$$

Furthermore, $f(x) \cdot g(x)$ is what you think it should be for $x \in (-R\ ,\ R)$. If $b_0 \neq 0$, then $\frac{f(x)}{g(x)}$ is what you think it should be but for only x sufficiently small enough.