Math 142.501 Prof. Girardi

Tue 1:00

Fall 98 Final Exam 12/07/98

NAME: Key

INSTRUCTIONS:

1. To receive credit you must:
   a. WORK IN A LOGICAL FASHION,
      SHOW ALL YOUR WORK,
      INDICATE YOUR REASONING.
   b. when applicable put your answer on/in the line/box provided
   c. if no such line/box is provided, then box your answer
   d. if you use your calculator, give an explanation of what you did on it.
2. There are 18 pages to this exam: a total of 15 equally weighted problems.
   Check that your copy of the exam has all of the pages.
3. During this test, do not leave your seat.
   If you have a question, raise your hand.
   When you finish: turn your exam over, put your pencil down, raise your hand.
4. This closed book/notes exam covers (from *Calculus*, by Varberg and Purcell) :
   § 7.1 – 7.7 , 8.1 – 8.6 , 9.1 – 9.4 , 10.1 – 10.3 , 11.1 – 11.8 , 12.6 – 12.8 .

Problem Source:

1. Fall 1998 Exam 1 # 1c
2. Fall 1998 Exam 1 # 3, simplified
3. Fall 1998 Exam 2 # 1a
4. Fall 1998 Exam 2 # 1c
5. Fall 1998 Exam 2 # 2a, simplified
6. Fall 1998 Exam 2 # 4c
7. Fall 1998 Exam 3 # 3d
8. Fall 1998 Exam 3 # 3b
9. Fall 1998 Exam 4 # 1a
10. Example from class
11. Similar to homework problems § 11.7 # 1-8 and examples from class
12. Similar to examples from class
13. Fall 1998 Exam 2 # 5
14. Fall 1998 Exam 3 # 4
15. An example from class: § 8.4 # 85

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bonus. the quote of our class!

BONUS:
Any fool can know, the point is to understand.
Albert Einstein

7(1

1. $\frac{d}{dx}\, e^{\tan x} =$ $(\sec^2 x)\ e^{\tan x}$.

2. $\frac{d}{dx}\,(\ln x)^{2x+3} = \underline{(\ln x)^{2x+3}\left[2\ln(\ln x) + \frac{2x+3}{x\ln x}\right]}$.

HINT: of the form $\frac{d}{dx}\left([f(x)]^{g(x)}\right)$.

$$y = (\ln x)^{2x+3}$$

$$\ln y = (2x+3)\,\ln(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\ln(\ln x) + (2x+3)\,\frac{1}{\ln x}\cdot\frac{1}{x}$$

3. $\int \sin^3(2x)\,dx =$ $-\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x$ $+ C$.

$$= \int (1-\cos^2 2x)\sin 2x\,dx$$

$$\boxed{\begin{array}{l} u = \cos 2x \\ du = -2\sin 2x\,dx \end{array}}$$

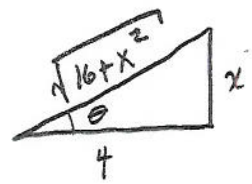
$$= -\frac{1}{2}\int (1-u^2)\,du$$

$$= -\frac{1}{2}\left[u - \frac{u^3}{3}\right] + c$$

4. $\displaystyle\int \frac{dx}{(16+x^2)^{3/2}} = \underline{\quad \frac{x}{16\,(16+x^2)^{1/2}} \quad} + \mathrm{C}$.

Your answer should NOT contain inverse trig functions ... convert back .

$$\boxed{\begin{aligned} x &= 4\tan\theta \\ dx &= 4\sec^2\theta\, d\theta \\ 16+x^2 &= 16\sec^2\theta \end{aligned}}$$



$$= \int \frac{4\sec^2\theta\, d\theta}{(4\sec\theta)^{2\cdot 3/2}} = \int \frac{4\sec^2\theta\, d\theta}{(4\sec\theta)^3}$$

$$= \frac{1}{4^2}\int \cos\theta\, d\theta = \frac{1}{16}\sin\theta \quad + C$$

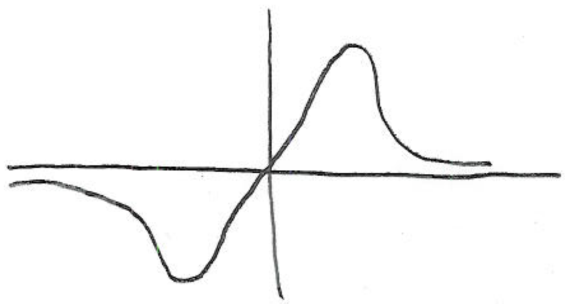
5. $\int \ln x \, dx =$ $x \ln x - x$ $+ C$.

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$\int \ln x \, dx = x \ln x - \int dx$

6. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx =$ DNE .



$$\int_0^b \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln|x^2+1|\Big|_0^b = \frac{1}{2}\ln|b^2+1| \to \infty$$

$$\int_a^0 \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln|x^2+1|\Big|_a^0 = -\frac{1}{2}|a^2+1| \to -\infty$$

7. $\lim_{n\to\infty}(-1)^n\dfrac{100n}{n^{3/2}+4} =$ ______ 0 ______.

8. $\lim_{n\to\infty}(-1)^n\dfrac{6n-5}{5n+1}$ = DNE .

9. $\sum_{n=7}^{\infty}(-1)^n \dfrac{n}{1+n^2}$

______ absolutely convergent

__✓__ conditionally convergent

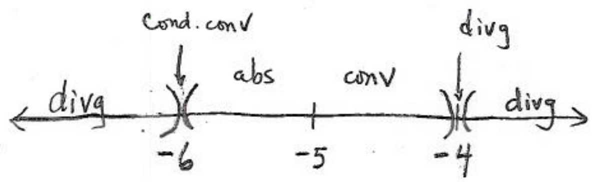
______ divergent

10. Indicate **precisely** where the series

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n}$$

is absolutely convergent, conditionally convergent, and divergent (a diagram as we did in class suffices).

$$\rho = \lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{n+1} \cdot \frac{n}{(x+5)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot |x+5| = |x+5| < 1$$



$$\sum \frac{(x+5)^n}{n} \underset{=}{x=-6} \sum \frac{(-1)^n}{n} \leftarrow \text{cond. conv.}$$

$$\sum \frac{(x+5)^n}{n} \underset{=}{x=-4} \sum \frac{1}{n} \leftarrow \text{divg}$$

11. Using the Geometric Series, find a power series representation for

$$f(x) = \frac{7}{9+4x^2}$$

and specify precisely when (i.e., for what values of x's?) this representation is valid.

$$\frac{7}{9+4x^2} = \frac{7}{9} \quad \frac{1}{1+\frac{4x^2}{9}}$$

$$= \frac{7}{9} \sum_{n=0}^{\infty} \left(\frac{-4x^2}{9}\right)^n$$

valid :

$$\left|\frac{4x^2}{9}\right| < 1$$

$$|x|^2 < \frac{9}{4}$$

$$\boxed{|x| < \frac{3}{2}}$$

12. Grandpa and Grandma Puffo are out for their morning walk over a time period of $0 \leqslant \theta \leqslant 2\pi$. Grandpa is walking along a path given by $r = \sin\theta$ while Grandma is walking along a path given by $r = \sqrt{(3\cos^2\theta)}$.

12a. Fill in the below chart, similar to as we did in class.

θ	$r = \sin\theta$	$r = \sqrt{(3\cos^2\theta)}$
$0 \to \pi/2$	$0 \xrightarrow{1} 1$	$\sqrt{3} \xrightarrow{5} 0$
$\pi/2 \to \pi$	$1 \xrightarrow{2} 0$	$0 \xrightarrow{6} \sqrt{3}$
$\pi \to 3\pi/2$	$0 \xrightarrow{3} -1$	$\sqrt{3} \xrightarrow{7} 0$
$3\pi/2 \to 2\pi$	$-1 \xrightarrow{4} 0$	$0 \xrightarrow{8} \sqrt{3}$

12b. Sketch below the curves of Grandpa and Grandma's walking paths, indicating parts 1–8, as we did in class

12c. GIVE <u>POINTS</u> IN POLAR FORM: (r, θ).
Grandpa and Grandma Puffo will pass eachother, and be able to give eachother

HIGH-FIVES, at time/times $\theta =$ $\pi/3$, $2\pi/3$

At this time/these times, they will be at the point/points $\left(\frac{\sqrt{3}}{2}, \pi/3\right)$ & $\left(\frac{\sqrt{3}}{2}, 2\pi/3\right)$.

Grandpa and Grandma Puffo will pass through the same place, but not at the same time (so they will not be able to give eachother HIGH-FIVES), at point/points

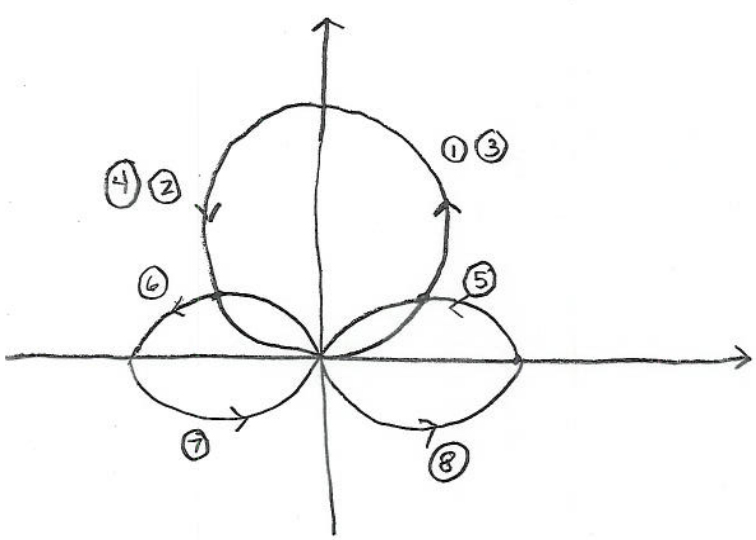
origin.

$\sin\theta = \sqrt{3\cos^2\theta}$

$\sin^2\theta = 3\cos^2\theta$

$\tan\theta = \sqrt{3} = \frac{\sqrt{3}/2}{1/2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \cancel{\frac{4\pi}{3}}, \cancel{\frac{5\pi}{3}}$

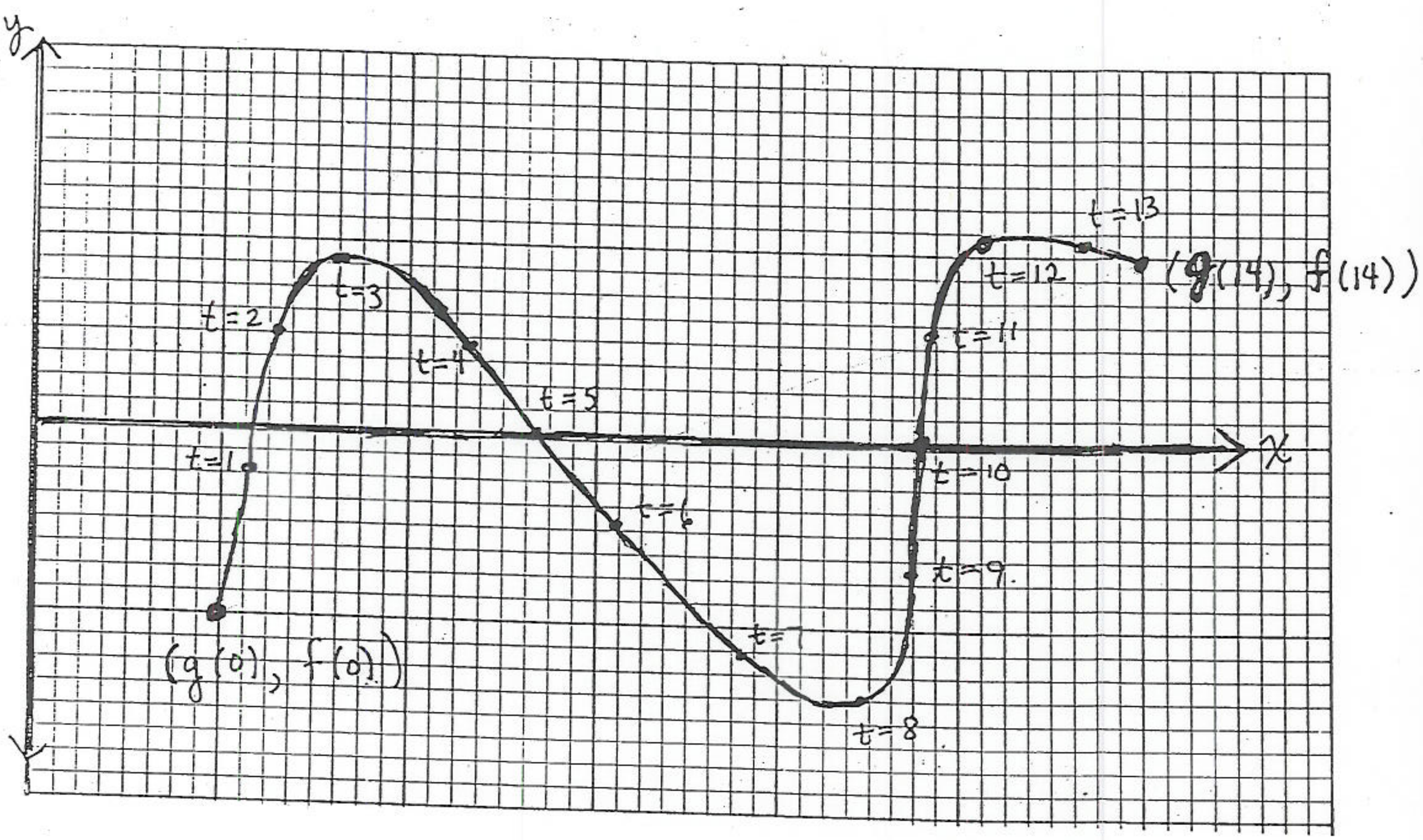


13. Let $f\colon [0,14] \to \mathbb{R}$ and $g\colon [0,14] \to \mathbb{R}$ be two continuous functions, each differentiable on $(0,14)$. Also $g'(x) \neq 0$ for each $x \in (0,14)$. Cauchy Puffetto is moving along the curve $\mathcal{C}$ show below in such a way that his position at time t, where $0 \leqslant t \leqslant 14$, is (x,y) where $x = g(t)$ and $y = f(t)$. Estimate all points c, where $c \in (0,14)$, for which

$$\frac{f(14)-f(0)}{g(14)-g(0)} = \frac{f'(c)}{g'(c)} = \frac{df/dt}{dg/dt} = \frac{dy}{dx}$$

ESTIMATE(S) ON THE C('S): 3, 8, 12

Clearly geometrically explain your reasoning! One point for the correct estimate(s) and 4 points for your explanation.



$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1}$$

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14. Formally show (i.e., by using the definition of convergence: page 514) that

$$\lim_{n\to\infty} \frac{2n^3}{n^3+1} = 2 .$$

Hints: I got you started, just fill in the missing parts. You might find helpful the observation that $\frac{2}{n^3+1} < \frac{2}{n^3}$.

Proof.

Recall that by definition of convergence of a sequence,

$$\lim_{n\to\infty} \frac{2n^3}{n^3+1} = 2$$

if and only if for each $\varepsilon > 0$ there exists a number $N_\varepsilon > 0$ such that if $n \geqslant N_\varepsilon$ then

$$\left| \frac{2n^3}{n^3+1} - 2 \right| < \varepsilon.$$

Let an arbitrary $\varepsilon > 0$ be given. Let $N_\varepsilon =$ ____________________. Then $n \geqslant N_\varepsilon$ implies that

$$\left| \frac{2n^3}{n^3+1} - 2 \right| = \left| \frac{2n^3 - 2(n^3+1)}{n^3+1} \right| = \frac{2}{n^3+1}$$

- $\frac{2}{n^3+1} < \frac{2}{n^3} \leq \frac{2}{N_\varepsilon^3} = \varepsilon \iff \sqrt[3]{\frac{2}{\varepsilon}} \equiv N_\varepsilon$

or

- $\frac{2}{n^3+1} \leq \frac{2}{N_\varepsilon^3+1} = \varepsilon \iff \sqrt[3]{\frac{2}{\varepsilon}-1} \equiv N_\varepsilon$

Thus, by the definition of convergence of a sequence,

$$\lim_{n\to\infty} \frac{2n^3}{n^3+1} = 2 .$$

■

15. Let $\mathbb{N}$ be the set of natural numbers, i.e.

$$\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}.$$

Consider a function

$$f\colon [-\pi, \pi] \to \mathbb{R}$$

whose derivative

$$f'\colon [-\pi, \pi] \to \mathbb{R}$$

is continuous. For $n \in \mathbb{N}$, defined the n^{th} Fourier coefficient of f to be

$$a_n \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

$u = f(x)$ $\quad dv = \sin(nx)\, dx$

$du = f'(x)\, dx$ $\quad v = \frac{-1}{n} \cos nx$

Show that $\lim_{n\to\infty} a_n = 0$. Hint: integration by parts.

$$a_n = \frac{1}{\pi} \left[-\frac{f(x) \cos nx}{n} \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \cos nx \, dx \right]$$

$$= \frac{-f(\pi) \cos n\pi}{\pi n} + \frac{f(-\pi) \cos n\pi}{\pi n} + \frac{\pi}{n} \int_{-\pi}^{\pi} f'(x) \cos nx \, dx$$

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More space for # 15: