MARK BOX		
PROBLEM	POINTS	
1 a–e	25	
2 a–c	15	
3 a–f	30	
4	5	
Total	75	
%	100	

Math 142.501 Prof. Girardi

Fall 98 Exam 3 11/17/98

NAME: Key

INSTRUCTIONS:

1. To receive credit you must:
   a. WORK IN A LOGICAL FASHION,
      SHOW ALL YOUR WORK,
      INDICATE YOUR REASONING.
   b. when applicable put your answer on/in the line/box provided
   c. if no such line/box is provided, then box your answer
   d. if you use your calculator, give an explanation of what you did on it.
2. The MARK BOX indicates the problems along with their points.
   Check that your copy of the exam has all of the problems.
3. Do not discuss this exam with anyone other than yourself.
4. This **OPEN** book/notes test covers (from *Calculus*, by Varberg and Purcell) : § 10.1 – 10.3, 11.1 .

Problem Source:

1. Inspired by look at problem § 10.6, Sample Test Problem # 8.
2. Similar to a Puffo problem from class.
3. Taken from my high-school calculus textbook.
4. Similar to an example from class and Example 1 from § 11.1.

BONUS: The first line of *The Carolinian Creed* is:

I will practice personal and academic integrity.

1. Numerical Integration with the function $f(x) = \ln x$. Let's approximate $\int_{0.8}^{1.2} \ln x \, dx$.

ps. my TI-83 says $\int_{.8}^{1.2} \ln x \approx -0.0026993$

1a. Fill in the below chart.

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}$
0	$\ln(x)$	0	0
1	x^{-1}	1	1
2	$-x^{-2}$	-1	$-\frac{1}{2}$
3	$2x^{-3}$	$2!$	$\frac{2!}{3!} = \frac{1}{3}$
4	$-3!\,x^{-4}$	$-3!$	$-\frac{3!}{4!} = -\frac{1}{4}$
5	$4!\,x^{-5}$	$4!$	$\frac{4!}{5!} = \frac{1}{5}$

1b. From the pattern in the chart, I see that for $n \geqslant 1$,

$$f^{(n)}(x) = \frac{(-1)^{n-1}\,(n-1)!}{x^n}$$

and

$$\frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n-1}}{n}$$

Hint: simplify the above as much as possible, it will make your life easier later on.

1c. For $n \geqslant 1$, the Taylor polynomial P_n of order n about $a = 1$ for $f(x) = \ln x$ is:

$$P_n(x) = \underline{1}\,(x-1)^1 + \underline{-\tfrac{1}{2}}\,(x-1)^2 + \underline{\tfrac{1}{3}}\,(x-1)^3 + \ldots + \underline{\tfrac{(-1)^{n-1}}{n}}\,(x-1)^n$$

and the order n Remainder Term is:

$$\ln x - P_n(x) \stackrel{\text{def}}{=} R_n(x) = \frac{(-1)^n \,(x-1)^{n+1}}{(n+1)\, c^{n+1}}$$

where c is between $\underline{\;x\;}$ and $\underline{\;1\;}$.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$= \frac{(-1)^n\, n!}{c^{n+1}} \cdot \frac{1}{(n+1)!} (x-1)^{n+1}$$

1d. From **1c**, we see that

$$\int_{0.8}^{1.2} \ln x \, dx \approx \int_{0.8}^{1.2} P_n(x) \, dx \qquad \textbf{(1d.1)}$$

where an upper bound on the error of approximation is:

$$\left| \int_{0.8}^{1.2} R_n(x) \, dx \right| . \qquad \textbf{(1d.2)}$$

Find n (as small as you can get it) so that the formula in **1d.2** insures that the approximation in **1d.1** is accurate to 4 decimal places.
Hint: $\int_{0.8}^{1.2} |x-1|^{n+1} \, dx = 2 \int_1^{1.2} (x-1)^{n+1} \, dx$.

$$\left| \int_{.8}^{1.2} R_n(x) \, dx \right|$$

$$\leq \int_{.8}^{1.2} |R_n(x)| \, dx$$

$$= \frac{1}{n+1} \int_{.8}^{1.2} \frac{|x-1|^{n+1}}{c^{n+1}} \, dx$$

$$\leq \frac{1}{n+1} \int_{.8}^{1.2} \frac{|x-1|^{n+1}}{(.8)^{n+1}} \, dx$$

$$= \frac{2}{(n+1)(.8)^{n+1}} \int_1^{1.2} (x-1)^{n+1} \, dx$$

$$= \frac{2}{(n+1)(.8)^{n+1}} \left. \frac{(x-1)^{n+2}}{n+2} \right|_1^{1.2}$$

$$= \frac{2}{(n+1)(n+2)} \frac{(.2)^{n+2}}{(.8)^{n+1}} = \frac{2}{(n+1)(n+2)} \frac{(.2)}{4^{n+1}} = \frac{2}{(n+1)(n+2)\, 5 \cdot 4 \cdot 4^n}$$

$$= \frac{1}{10(n+1)(n+2)\, 4^n} \overset{\text{WANT}}{\leq} .00005$$

up

n	$\frac{1}{10(n+1)(n+2)4^n}$	
3	$\frac{1}{10 \cdot 4 \cdot 5 \cdot 4^3}$	$\approx .000078$
4	$\frac{1}{10 \cdot 5 \cdot 6 \cdot 4^4}$	$\approx .0000132$ 60208

$$\boxed{n = 4}$$

1e. Using a Talyor polynomial, approximate $\int_{0.8}^{1.2} \ln x \, dx$ to 4 decimal places.

$\int_{0.8}^{1.2} \ln x \, dx \approx$ $-.0026$.

$$\int_{.8}^{1.2} \ln x \, dx \overset{4dp}{\approx} \int_{.8}^{1.2} P_4(x) \, dx$$

$$= \int_{.8}^{1.2} \left[(x-1)^1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 \right] dx$$

$$= \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{20} \Bigg|_{.8}^{1.2}$$

$$= \left[\frac{(.2)^2}{2} - \frac{(.2)^3}{6} + \frac{(.2)^4}{12} - \frac{(.2)^5}{20} \right]$$

$$- \left[\frac{(.2)^2}{2} - \frac{-(.2)^3}{6} + \frac{(.2)^4}{12} - \frac{-(.2)^5}{20} \right]$$

$$= -2 \left[\frac{(.2)^3}{6} + \frac{(.2)^5}{20} \right]$$

$$= -2 \left[\frac{1}{5^3 \cdot 6} + \frac{1}{5^5 \cdot 20} \right]$$

$$= -\frac{2}{2 \cdot 5^3} \left[\frac{1}{3} + \frac{1}{250} \right] = \frac{-1}{5^3} \left[\frac{250+3}{750} \right]$$

$$= \frac{-253}{5^3 \cdot 5^3 \cdot 6} = \frac{-253}{5^6 \cdot 6} \approx \underbrace{-.0026}_{\text{take this much.}}98.6667$$

…e that (just for fun)

$$-.0026986667 - .0000130208 \le \int_{.8}^{1.2} \ln x \, dx \le -.0026986667 + .0000130208$$

$$\| \qquad\qquad\qquad \text{S||\ TI-83} \qquad\qquad\qquad \|$$

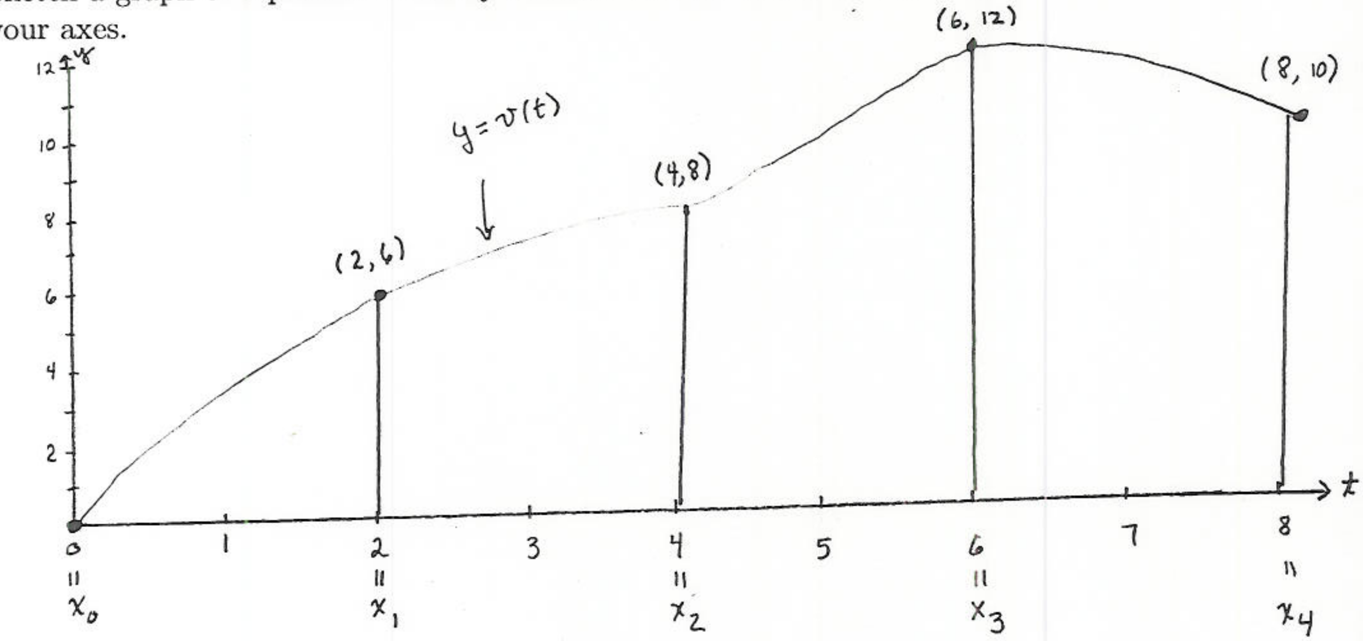
$$-.0027116875 \le -.0026993 \le -.0026856459$$

2. Mr. Puffo is moving along the real number line. He observes his velocity at some key times as:

t	0	2	4	6	8
$v(t)$	0	6	8	12	10

The units are minutes and feet.

2a. Sketch a graph of a possible velocity function for Mr. Puffo. Do do forget to label your axes.



$$\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$$

distance $= \int v(t)\,dt$

2b. The Trapezoidal Rule (with $n = 4$) gives that Mr. Puffo travel approximately

62 feet in these 8 minutes.

$$T_4 = \frac{\Delta x}{2}\left[v(x_0) + 2v(x_1) + 2v(x_2) + 2v(x_3) + 1v(x_4)\right]$$

$$= \frac{2}{2}\left[1\cdot v(0) + 2v(2) + 2v(4) + 2v(6) + 1\cdot v(8)\right]$$

$$= 1\left[1\cdot 0 + 2\cdot 6 + 2\cdot 8 + 2\cdot 12 + 1\cdot 10\right]$$

$$= 0 + 12 + 16 + 24 + 10$$

$$= 62$$

2c. The Parabolic Rule (with $n = 4$) gives that Mr. Puffo travel approximately

65 ⅓ feet in these 8 minutes.

$$P_4 = \frac{\Delta x}{3}\left[1\cdot v(x_0) + 4v(x_1) + 2v(x_2) + 4v(x_3) + 1v(x_4)\right]$$

$$= \frac{2}{3}\left[1\cdot v(0) + 4v(2) + 2v(4) + 4v(6) + 1v(8)\right]$$

$$= \frac{2}{3}\left[1\cdot 0 + 4\cdot 6 + 2\cdot 8 + 4\cdot 12 + 1\cdot 10\right]$$

$$= \frac{2}{3}\left[24 + 16 + 48 + 10\right]$$

$$= \frac{2\cdot 98}{3} = \frac{196}{3} = 65\frac{1}{3}$$

3. Determine if the limits of the following sequences exist. If your answer is DNE, explain why.

3a. $\lim_{n\to\infty} \dfrac{6n-5}{5n+1} = \underline{\quad 6/5 \quad}$.

$$\lim_{x\to\infty} \frac{6x-5}{5x+1} \overset{\text{Algebra}}{=} \lim_{x\to\infty} \frac{6-\frac{5}{x}}{5+\frac{1}{x}} = \frac{6}{5}$$

3b. $\lim_{n\to\infty} (-1)^n \dfrac{6n-5}{5n+1} = \underline{\text{DNE - oscillating btw. } \approx +\tfrac{6}{5} \text{ \& } \approx -\tfrac{6}{5}}$.

3c. $\lim_{n\to\infty} \dfrac{100n}{n^{3/2}+4} =$ $\underline{\quad 0 \quad}$.

$$\lim_{x\to\infty} \frac{100x}{x^{3/2}+4} = \lim_{x\to\infty} \frac{\frac{100}{x^{1/2}}}{1+\frac{4}{x^{3/2}}} = \frac{0}{1+0}$$

3d. $\lim_{n\to\infty} (-1)^n \dfrac{100n}{n^{3/2}+4} =$ $\underline{\quad 0 \quad}$.

$$-\frac{100n}{n^{3/2}+4} \le (-1)^n \frac{100n}{n^{3/2}+4} \le +\frac{100n}{n^{3/2}+4}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad\qquad 0$$

Squeeze Theorem.

3e. $\lim_{n\to\infty} \dfrac{\sin n}{n} = $ 0 .

$$\begin{array}{ccccc} \dfrac{-1}{n} & \leq & \dfrac{\sin n}{n} & \leq & \dfrac{1}{n} \\ \Big\downarrow {\scriptstyle n\to\infty} & & & & \Big\downarrow {\scriptstyle n\to\infty} \\ 0 & & & & 0 \end{array}$$

Squeeze Theorem

3f. $\lim_{n\to\infty} \sin(n\pi) = $ 0 .

note $\sin(n\pi) = 0 \quad \forall\ n \in \mathbb{N}$.

4. Formally show (i.e., by using the definition of convergence: page 514) that

$$\lim_{n\to\infty} \frac{2n^3}{n^3+1} = 2 .$$

Hints: I got you started, just fill in the missing parts. You might find helpful the observation that $\frac{2}{n^3+1} < \frac{2}{n^3}$.

Proof.

Recall that by definition of convergence of a sequence,

$$\lim_{n\to\infty} \frac{2n^3}{n^3+1} = 2$$

if and only if for each $\varepsilon > 0$ there exists a number $N_\varepsilon > 0$ such that if $n \geqslant N_\varepsilon$ then

$$\left| \frac{2n^3}{n^3+1} - 2 \right| < \varepsilon.$$

Let an arbitrary $\varepsilon > 0$ be given. Let $N_\varepsilon = \sqrt[3]{\frac{2}{\varepsilon}}$. Then $n \geqslant N_\varepsilon$ implies that

$$\left| \frac{2n^3}{n^3+1} - 2 \right| = \left| \frac{2n^3 - 2(n^3+1)}{n^3+1} \right|$$

$$= \left| \frac{-2}{n^3+1} \right|$$

$$= \frac{2}{n^3+1}$$

$$< \frac{2}{n^3} \underset{N_\varepsilon \leq n}{\leq} \frac{2}{N_\varepsilon^3} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon .$$

Thus, by the definition of convergence of a sequence,

$$\lim_{n\to\infty} \frac{2n^3}{n^3+1} = 2 .$$

■