MARK BOX		
PROBLEM	POINTS	
1 a–e	25	
2 a–c	15	
3 a–c	15	
4 a–c	15	
5	5	
Total	75	
%	100	

Math 142.501 Prof. Girardi

Fall 98 Exam 2 10/29/98

NAME: Answer Key

INSTRUCTIONS:

1. To receive credit you must:
   a. WORK IN A LOGICAL FASHION,
      SHOW ALL YOUR WORK,
      INDICATE YOUR REASONING.
   b. when applicable put your answer on/in the line/box provided
   c. if no such line/box is provided, then box your answer
   d. if you use your calculator, give an explanation of what you did on it.
2. The MARK BOX indicates the problems along with their points.
   Check that your copy of the exam has all of the problems.
3. During this test, do not leave your seat.
   If you have a question, raise your hand.
   When you finish: turn your exam over, put your pencil down, raise your hand.
4. This closed book/notes quiz covers (from *Calculus*, by Varberg and Purcell) :
   § 8.1 – 8.6 , 9.1 – 9.4 .

Problem Source:

1. look at problems § 8.6 sample test # 's 6, 37, 42, 41
   & look at problem § 8.? # 57

2a. look at problem § 8.? # 64 , also, an example from class

2b. look at problem § 8.? # 45 , also, an example from class

2c. look at problem § 9.4 # 44

3. look at problems: § 9.5 sample test #'s 7 & 15. an example from class.
4. look at problems: § 9.5 sample test #'s 31 & 33, § 9.3 # 17
5. concept comprehension indicator — suggested to me by a senior math major

Any fool can <u>know</u>, the point is to <u>understand</u>.

<u>Albert</u> Einstein.

1. Five integrals: **1a – 1e** . ⊛ HINT: + C

1a. $\int \sin^3(2x)\,dx = -\frac{1}{2}\cos(2x) + \frac{1}{6}\cos^3(2x) + C$.

$$\int \sin^3(2x)\,dx = \int \sin^2(2x)\,\sin(2x)\,dx = \int [1-\cos^2(2x)]\,\sin(2x)\,dx$$

$$= -\frac{1}{2}\int (1-u^2)\,du = -\frac{1}{2}\left[u - \frac{u^3}{3}\right] + C$$

$\boxed{u = \cos(2x) \quad du = -2\sin(2x)\,dx}$

WAY # 3

$u = x \qquad dv = (x+5)^{-1/2}\,dx$

$du = dx \qquad v = 2(x+5)^{1/2}$

$$\int \frac{x\,dx}{\sqrt{x+5}} = 2x(x+5)^{1/2} - 2\int (x+5)^{1/2}\,dx$$

$$= 2x(x+5)^{1/2} - \frac{4}{3}(x+5)^{3/2} + C$$

⊛ to here is fine but

Note

$$= \frac{2}{3}(x+5)^{1/2}\,(3x - 2(x+5)) + C$$

$$= \frac{2}{3}(x+5)^{1/2}\,(x-10) + C$$

1b. $\int \frac{x}{\sqrt{x+5}}\,dx = \frac{2}{3}(x+5)^{3/2} - 10(x+5)^{1/2} + C$

note $\frac{2}{3}(x+5)^{1/2}\,[\underbrace{(x+5) - 15}_{(x-10)}] + C$

WAY # 1

$u = x+5$

$du = dx$

$$\int \frac{x}{\sqrt{x+5}}\,dx = \int \frac{u-5}{u^{1/2}}\,du = \int (u^{1/2} - 5u^{-1/2})\,du$$

$$= \frac{2}{3}u^{3/2} - 10u^{1/2} + C$$

WAY # 2

$u = (x+5)^{1/2}$

$du = \frac{1}{2}(x+5)^{-1/2}\,dx$

$u^2 = x+5$

$$\int \frac{x}{\sqrt{x+5}}\,dx = 2\int (u^2 - 5)\,du$$

$$= 2\left[\frac{u^3}{3} - 5u\right] + C$$

$$= \frac{2}{3}u^3 - 10u + C$$

1c. $\displaystyle\int \frac{dx}{(16+x^2)^{3/2}} = \frac{x}{16\sqrt{x^2+16}} + C$

$x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$16+x^2 = 16 + 16\tan^2\theta$

$= 16(1+\tan^2\theta)$

$= 16\sec^2\theta$

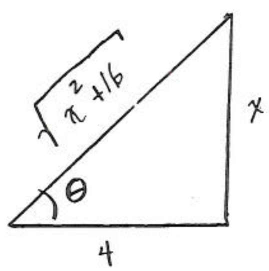
$(16+x^2)^{3/2} = (16\sec^2\theta)^{3/2}$

$= (4^2)^{3/2}\left[(\sec\theta)^2\right]^{3/2}$

$= 4^3\sec^3\theta$

$$\int \frac{dx}{(16+x^2)^{3/2}} = \int \frac{4\sec^2\theta\, d\theta}{4^3\sec^3\theta} = \frac{1}{4^2}\int \cos\theta\, d\theta$$

$$= \frac{1}{16}\sin\theta + C$$



$\Rightarrow \sin\theta = \dfrac{x}{\sqrt{x^2+16}}$

1d. $\displaystyle\int \frac{4x^2+3x+6}{x^2(x^2+3)}\,dx = \underline{\ln|x| - 2x^{-1} - \frac{1}{2}\ln|x^2+3| + \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$

$$\frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}}$$

$$\frac{4x^2+3x+6}{\underbrace{(x-0)^2}(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3} = \frac{A(x)(x^2+3) + B(x^2+3) + (Cx+D)x^2}{x^2(x^2+3)}$$

a squared linear term, <u>not</u> an irreducible quadratic

$$\Rightarrow \quad 4x^2+3x+6 = Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2 \quad \boxed{x=0} \Rightarrow 6 = 3B \Rightarrow \boxed{B=2}$$

x^3 : $0 = A + C$ $\Rightarrow \boxed{C=-1}$

x^2 : $4 = B + D \Rightarrow \boxed{D=2}$

x^1 : $3 = 3A \Longrightarrow \boxed{A=1}$

x^0 : $6 = 3B$

$$\int \frac{4x^2+3x+6}{x^2(x^2+3)}\,dx = \int \left[x^{-1} + 2x^{-2} + \frac{-x}{x^2+3} + \frac{2}{x^2+3} \right] dx$$

and

$$\int \frac{2}{x^2+3}\,dx = \int \frac{2/3}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx = \frac{2}{3}\sqrt{3}\int \frac{du}{u^2+1}$$

$$= \frac{2\sqrt{3}}{3}\tan^{-1} u + C$$

$$\boxed{\begin{aligned} u &= \frac{x}{\sqrt{3}} \\ \sqrt{3}\,du &= dx \end{aligned}}$$

1e. Let a and b be nonzero constants.

$$\int e^{ax} \sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right] + C.$$

$u = e^{ax}$	$dv = \sin(bx)\,dx$	$\tilde{u} = e^{ax}$	$d\tilde{v} = \cos(bx)\,dx$
$du = a e^{ax}\,dx$	$v = -\frac{1}{b}\cos(bx)$	$d\tilde{u} = a e^{ax}\,dx$	$\tilde{v} = \frac{1}{b}\sin(bx)$

$$\int e^{ax}\sin(bx)\,dx \overset{u}{\underset{dv}{=}} -\frac{e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos(bx)\,dx$$

$$\overset{\tilde{u}}{\underset{d\tilde{v}}{=}} -\frac{e^{ax}}{b}\cos bx + \frac{a}{b}\left[\frac{e^{ax}}{b}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx\right]$$

$$= \frac{-e^{ax}\cos bx}{b} + \frac{a e^{ax}\sin bx}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

<u>So.</u>

$$\underbrace{\left(1+\frac{a^2}{b^2}\right)}_{=\frac{b^2+a^2}{b^2}}\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{b^2}\left[a\sin bx - b\cos bx\right] + C$$

<u>So.</u>

$$\int e^{ax}\sin bx\,dx = \frac{b^2}{b^2+a^2}\cdot\frac{e^{ax}}{b^2}\left[a\sin bx - b\cos bx\right] + C$$

WAY # 2

$u = \sin(bx)$ $\quad dv = e^{ax}\,dx$ $\quad \tilde{u} = \cos(bx)$ $\quad d\tilde{v} = e^{ax}\,dx$

2. Fun with Reduction Formulae **2a – 2c** .

2a. Let α be a constant. Clearly show that

$$\int (\ln x)^\alpha \, dx = x(\ln x)^\alpha - \alpha \int (\ln x)^{\alpha-1} \, dx .$$

$u = (\ln x)^\alpha$	$dv = dx$
$du = \alpha (\ln x)^{\alpha-1} \cdot \frac{1}{x} \, dx$	$v = x$

Using integration by parts :

$$\int (\ln x)^\alpha \, dx = x(\ln x)^\alpha - \alpha \int (\ln x)^{\alpha-1} \, dx$$

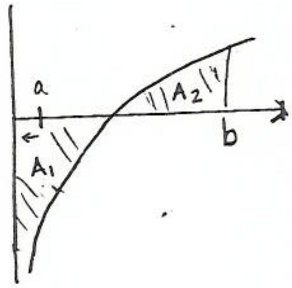
2b. Using **2a**, I see that $\int (\ln x)^2\, dx = \underline{x(\ln x)^2 - 2x\ln x + 2x + C}$.

$$\int (\ln x)^2\, dx \overset{\alpha=2}{\underset{(2a)}{=\!=}} x(\ln x)^2 - 2\int (\ln x)^1\, dx$$

$$\overset{\alpha=1}{\underset{(2a)}{=\!=}} x(\ln x)^2 - 2\left[x(\ln x)^1 - 1\int (\ln x)^0\, dx \right]$$

$$= x(\ln x)^2 - 2x\ln x + 2\int 1\, dx$$

2c. Find all real numbers b so that $\int_0^b \ln x\, dx = 0$. ANSWER: e.



want $|A_1| = |A_2|$.

Note : b must be ≥ 0.

$$2a \Rightarrow \int \ln x \, dx \stackrel{\alpha=1}{=\!=} x \ln x - \int 1 \, dx = x \ln x - x + C$$

$$\int_0^b \ln x \, dx = \lim_{a \to 0^+} \int_a^b \ln x \, dx = \lim_{a\to 0^+} \left[x \ln x - x \Big|_a^b \right]$$

$$= \lim_{a \to 0^+} \left[b \ln b - b - a \ln a + a \right]$$

$$= b \ln b - b - \lim_{a \to 0^+} \frac{\ln a}{a^{-1}} + \lim_{a \to 0^+} a$$

$$= b \ln b - b - \lim_{a \to 0^+} \frac{a^{-1}}{-a^{-2}} + 0$$

$$= b \ln b - b + \lim_{a \to 0^+} a$$

$$= b \ln b - b$$

AND. $b \ln b - b = 0 \iff b(\ln b - 1) = 0$

$\iff b = 0$ or $\ln b = 1$

$b = 0$: not in the domain of the integrand.

$\ln b = 1 \Updownarrow b = e$

3. Three L'Hôpital-ers **4a – 4c** .

3a. $\lim_{x\to\infty} \frac{\ln x}{x^2} =$ 0 .

$$\lim_{x\to\infty} \frac{\ln x}{x^2} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{x^{-1}}{2x^1} = \frac{1}{2} \lim_{x\to\infty} \frac{1}{x^2} = 0$$

3b. $\lim_{x\to\infty} \left(\frac{1}{\sin x} - \frac{1}{x}\right) =$ DNE – oscillates. .

$\lim_{x\to\infty} \frac{1}{\sin x}$ DNE since it oscillates

$\lim_{x\to\infty} \frac{1}{x} = 0$

So $\lim_{x\to\infty} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ DNE since it oscillates.

P.S. I copied the problem wrong – should have been $\lim_{x\to 0}$

3c. Let c be a constant. $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = \underline{\quad e^c \quad}$.

REMARK: Even if you remember the answer, show the work!

$$y = \left(1+\frac{c}{x}\right)^x$$

$$\ln y = x \ln\left(1+\frac{c}{x}\right)$$

$$\lim_{x\to\infty} x \ln\left(1+\frac{c}{x}\right) \underset{\text{algebra}}{\overset{\infty\cdot 0}{=\!=}} \lim_{x\to\infty} \frac{\ln\left(\frac{x+c}{x}\right)}{\frac{1}{x}}$$

$$\underset{\text{L'H}}{\overset{\frac{0}{0}}{=\!=}} \lim_{x\to\infty} \frac{\frac{x}{x+c}\cdot c\, d_x \frac{1}{x}}{d_x\, 1/x}$$

$$\underset{=\!=}{\text{algebra}}\; c \lim_{x\to\infty} \frac{x}{x+c}$$

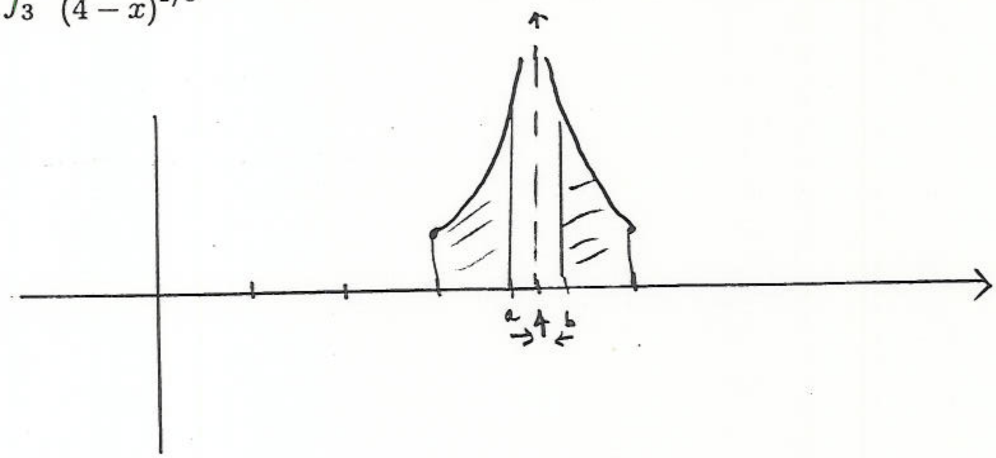
$$\underset{=\!=}{\text{algebra}}\; c \lim_{x\to\infty} \frac{1}{1+\frac{c}{x}} = c\cdot\frac{1}{1+0} = c$$

So $\ln y \to c$

so $y = e^{\ln y} \to e^c$

4. Three Improper Integrals 4a – 4c
If your answer is DOES NOT EXIST, specifically explain why.

4a. $\displaystyle\int_3^5 \frac{dx}{(4-x)^{2/3}} =$ 6.



$$\int_3^5 \frac{dx}{(4-x)^{2/3}} = \lim_{a\to 4^-} \int_3^a (4-x)^{-2/3}\,dx + \lim_{b\to 4^+} \int_b^5 (4-x)^{-2/3}\,dx$$

$$= \lim_{a\to 4^-} -3(4-x)^{1/3}\Big|_3^a + \lim_{b\to 4^+} -3(4-x)^{1/3}\Big|_b^5$$

$$= +3\lim_{a\to 4^-} (4-x)^{1/3}\Big|_a^3 + 3\lim_{b\to 4^+} (4-x)^{1/3}\Big|_5^b$$

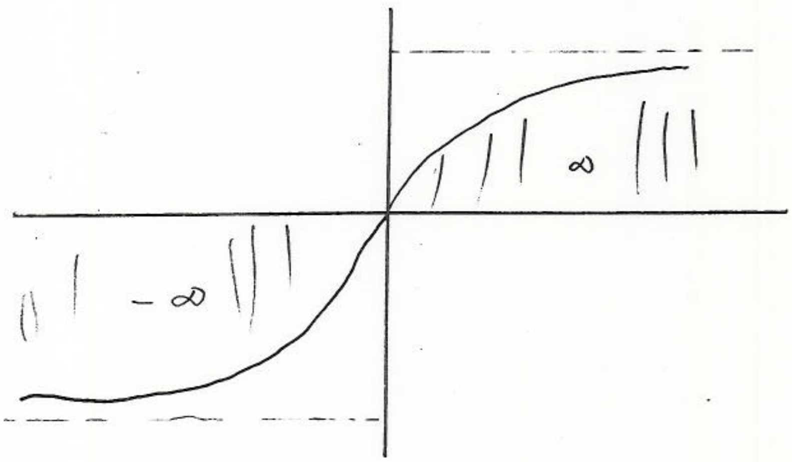
$$= 3\lim_{a\to 4^-}\left[1-(4-a)^{1/3}\right] + 3\lim_{b\to 4^+}\left[(4-b)^{1/3} - -1\right]$$

$$= 3\cdot 1 + 3\cdot 1$$

$$= 6$$

4b. $\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+9}}\,dx =$ DNE $\infty - \infty$.

Note $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+9}} \overset{\text{algebra}}{=} \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{9}{x^2}}} = 1$



just look at graph!

no calculus needed!

be clever.

If you are not clever, then you got to work....

$$\int_0^\infty \frac{x}{\sqrt{x^2+9}}\,dx = \lim_{b\to\infty} \int_0^b (x^2+9)^{-1/2}\,x\,dx$$

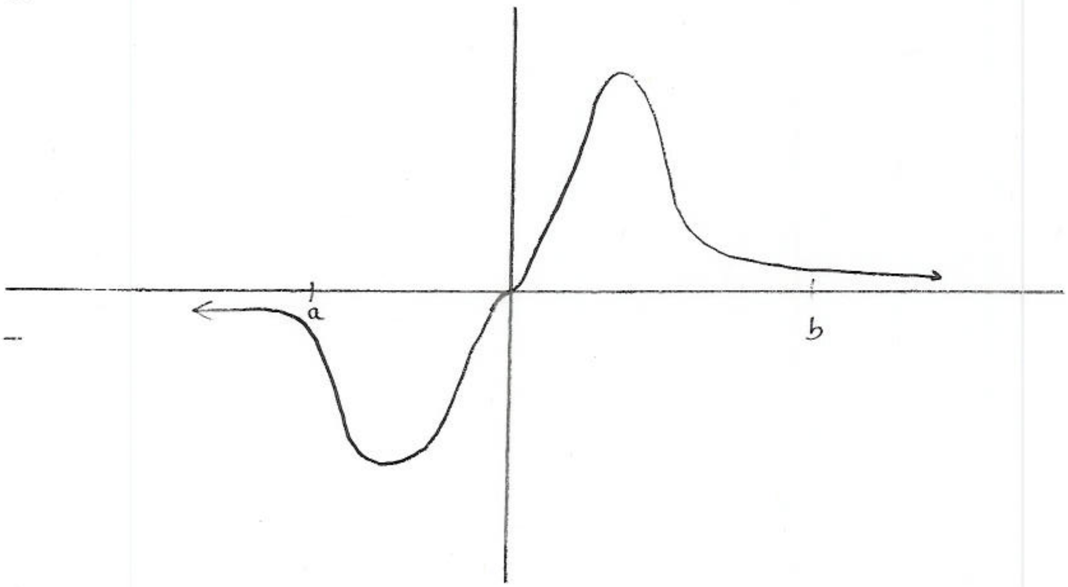
$$= \lim_{b\to\infty} (x^2+9)^{1/2} \Big|_0^b$$

$$= \lim_{b\to\infty} \sqrt{b^2+9} - 3 = \infty$$

by symmetry

$$\int_{-\infty}^0 \frac{x\,dx}{\sqrt{x^2+9}} = -\infty.$$

4c. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx =$ ______________________________.



$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^2+1} = \lim_{a \to -\infty} \int_a^0 \frac{x\,dx}{x^2+1} + \lim_{b\to\infty} \int_0^b \frac{x\,dx}{x^2+1}$$

$\underline{\underline{\text{see}}}$ below $\quad -\infty \quad + \quad \infty \qquad$ DNE

$$\lim_{b\to\infty} \int_0^b \frac{x\,dx}{x^2+1} = \lim_{b\to\infty} \frac{1}{2} \ln|x^2+1| \Big|_0^b$$

$$= \frac{1}{2} \lim_{b\to\infty} \ln(b^2+1) - 0 = \infty$$

by symmetry:

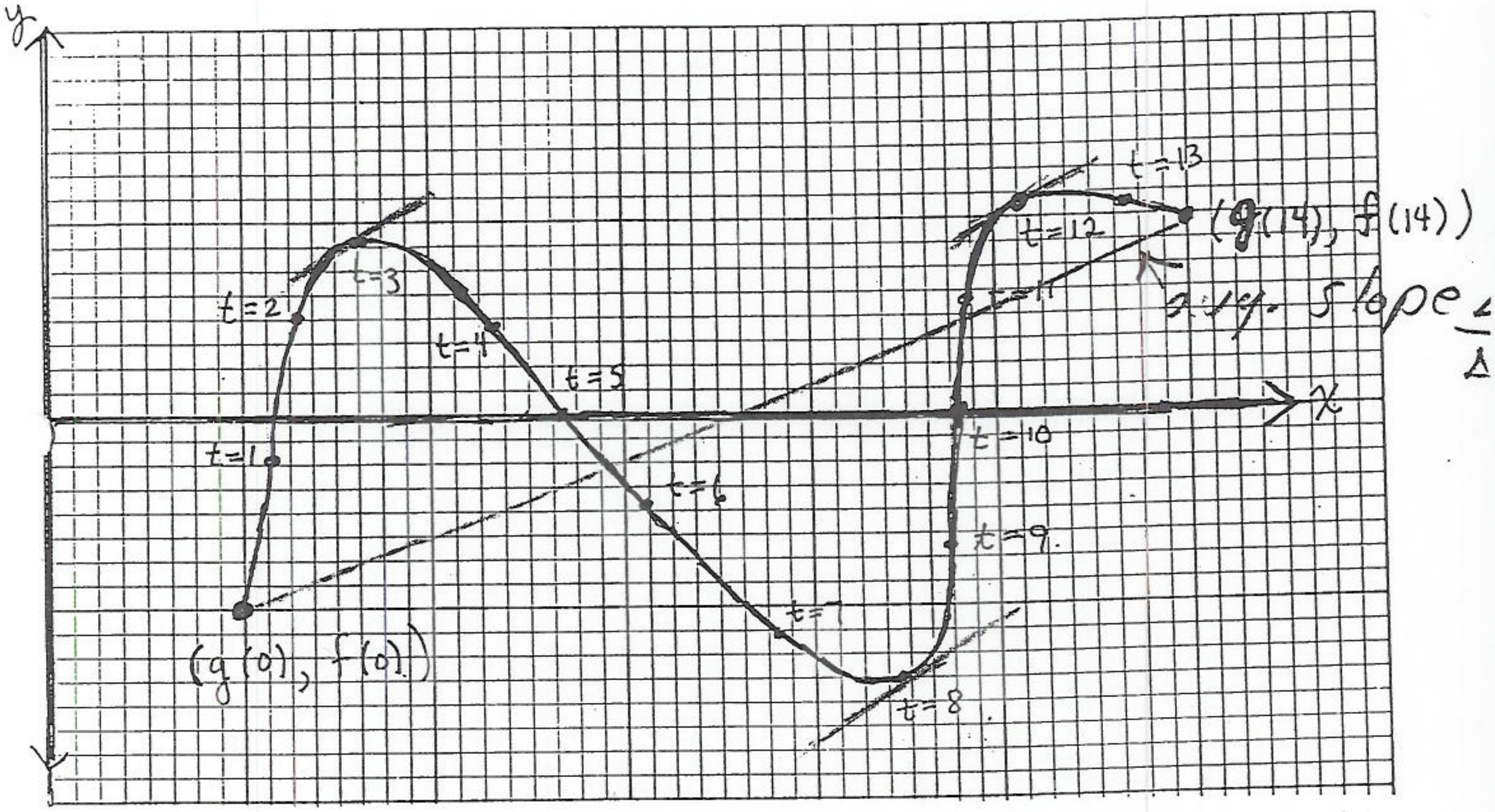
$$\lim_{a\to-\infty} \int_a^0 \frac{x\,dx}{x^2+1} = -\infty.$$

5. Let $f\colon [0,14] \to \mathbb{R}$ and $g\colon [0,14] \to \mathbb{R}$ be two continuous functions, each differentiable on $(0,14)$. Also $g'(x) \neq 0$ for each $x \in (0,14)$. Cauchy Puffetto is moving along the curve C show below in such a way that his position at time t, where $0 \leqslant t \leqslant 14$, is (x,y) where $x = g(t)$ and $y = f(t)$. Estimate all points c, where $c \in (0,14)$, for which

$$\frac{f(14)-f(0)}{g(14)-g(0)} = \frac{f'(c)}{g'(c)}.$$

ESTIMATE(S) ON THE C('S): 3, 8, 12.

Clearly geometrically explain your reasoning! One point for the correct estimate(s) and 4 points for your explanation.



$g'(x) \neq 0$ means that he never goes left

$f'(t) = \frac{dy}{dt}$ $\quad g'(t) = \frac{dx}{dt}$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$ = slope $\quad \frac{f(14)-f(0)}{g(14)-g(0)}$ = average slope

more space provided ⟹ over.

More space for # 5:

$$\text{average slope} = \frac{\Delta y}{\Delta x} \quad \frac{f(t_f) - f(t_i)}{g(t_f) - g(t_i)}$$

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)}$$

the question is asking for when is the tangent line parallel to the line crossing pts.

$(g(0), f(0))$ and $(g(14), f(14))$

by looking at the graph, these appear to be at $2 < t < 3$, $t = 8$, $t = 12$.