MATH 142 sections 004 & 005
FALL 1993 EXAM # 2

1. Let c be a fixed constant. Find the limit of the following sequence. Work carefully & indicate the reasoning behind your answer.

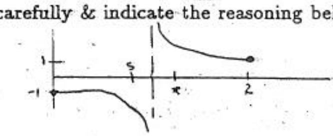
a) $\lim_{n\to\infty} [1+\frac{c}{n}]^n = \boxed{e^c}$

Indeterminate form 1^∞ so will need to $\ln$ & L'H the corresp. func. $y=(1+\frac{c}{x})^x$.

$\ln[1+\frac{c}{x}]^x = \lim_{x\to\infty} \frac{\ln[1+\frac{c}{x}]}{\frac{1}{x}} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{x}{x+c}\cdot c\cdot D_x(x^{-1})}{D_x(x^{-1})} = c \lim_{x\to\infty} \frac{x}{x+c} = c\cdot 1.$

2. Evaluate the following integral. Work carefully & indicate the reasoning behind your answer.

$$\int_0^2 \frac{dx}{(x-1)^3} = \boxed{DNE}$$



$\int_0^2 (x-1)^{-3}\,dx = \left[\lim_{s\to 1^-} \int_0^s (x-1)^{-3}\,dx\right] + \left[\lim_{t\to 1^+} \int_t^2 (x-1)^{-3}\,dx\right] = \text{"}-\infty+\infty\text{"}$ ← indeterminate form.

$\int_0^s (x-1)^{-3}\,dx = -\frac{1}{2}(x-1)^{-2}\Big|_0^s = -\frac{1}{2}\frac{1}{(s-1)^2} + \frac{1}{2} \xrightarrow{s\to 1^-} -\infty + \frac{1}{2} = -\infty$

$0<s<1$

$1<t<2$

$\int_t^2 (x-1)^{-3}\,dx = -\frac{1}{2}(x-1)^{-2}\Big|_t^2 = -\frac{1}{2} + \frac{1}{2}\frac{1}{(t-1)^2} \xrightarrow{t\to 1^+} -\frac{1}{2}+\infty = \infty$

4. Mr. Taylor

a) Find the 3rd degree Taylor polynomial $P_3(x)$ and the remainder term $R_3(x)$ about the point $a=1$ for the function $f(x)=\ln x$.

Answer: $P_3(x) = \boxed{(x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$

$R_3(x) = \boxed{-\frac{1}{4}\frac{1}{z^4}(x-1)^4}$ where z is between $\boxed{x \,\&\, 1}$.

$f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f^3(x) = 2x^{-3}$ $\quad f^3(1) = 2$

$f^4(x) = -2\cdot 3x^{-4}$

$\Rightarrow P_3(x) = 0 + \frac{1}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$

$R_3(x) = \frac{-2\cdot 3 z^{-4}}{4!}(x-1)^4$

For $.8 \le x \le 1.2$:

$|R_3(x)| = \frac{1}{4}\frac{1}{|z|^4}|x-1|^4 \le \frac{1}{4}\left(\frac{1}{.8}\right)^4 (.2)^4 = \frac{1}{4}\left(\frac{1}{4}\right)^4 = \frac{1}{4^5} = \frac{1}{1024} \sim .000976$

b) In part (a), to how many decimal places of accuracy does Taylor's formula guarantee that $P_3(x)$ approximate $f(x)=\ln x$ for x between .8 and 1.2?

Answer: $\boxed{2}$ decimal places of accuracy

3. Determine whether each of the following 4 series is absolutely convergent, conditionally convergent, or divergent. CLEARLY explain your reasoning and indicate the test(s) used. No credit will be given the work that does not make sense to us!!!!

a) $\sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n$

$\Sigma\,|(-\frac{2}{3})^n| = \Sigma(\frac{2}{3})^n$ ← geometric series. $r=\frac{2}{3}$, $|r|<1$

✓ absolutely convergent
___ conditionally convergent
___ divergent

b) $\sum_{n=1}^{\infty}\frac{(-2)^n}{3^n-1}$

$|a_n| = \frac{2^n}{3^n-1} \le \frac{2^n}{(\frac{2}{3})3^n} = \frac{3}{2}\cdot\left(\frac{2}{3}\right)^n$.

$\Sigma(\frac{2}{3})^n$ conv. (geom. series w/ $r=\frac{2}{3}$), ∴ by Comparison Test, $\Sigma|a_n|$ conv.

✓ absolutely convergent
___ conditionally convergent
___ divergent

c) $\sum_{n=1}^{\infty}(-1)^n\frac{n^{\frac{1}{3}}}{n+1}$

___ absolutely convergent
X conditionally convergent
___ divergent

1st Examine $\Sigma\frac{n^{1/3}}{n+1}$. LCT w/ $b_n = \frac{n^{1/3}}{n^1} = \frac{1}{n^{2/3}}$

$\frac{a_n}{b_n} = \frac{n^{\frac{1}{3}}}{n+1}\cdot\frac{n^{\frac{2}{3}}}{1} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \xrightarrow{n\to\infty} 1 \;\; 0<1<\infty$

Σb_n divg -- p-series, $p=2/3<1 \Rightarrow \Sigma\frac{n^{1/3}}{n+1}$ divg.

2nd Alt. Series Test w/ $a_n = \frac{n^{1/3}}{n+1} > 0$

• a_n decreasing? $f(x) = \frac{x^{\frac{1}{3}}}{x+1} \Rightarrow f'(x) = \frac{\frac{1}{3}x^{-\frac{2}{3}}(x+1) - x^{\frac{1}{3}}(1)}{(x+1)^2} = \frac{\frac{1}{3}x^{-2/3}[x+1-3x]}{(\quad)^2} = \frac{\frac{1}{3}x^{-2/3}[1-2x]}{(\quad)^2}$

yes since $f'(x)<0$.

• $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^{1/3}}{n+1} = \lim_{n\to\infty}\frac{\frac{1}{n^{2/3}}}{1+\frac{1}{n^{1/3}}} = \frac{0}{1+0} = 0.$

∴ by Alt. Series Test. $\Sigma(-1)^n\frac{n^{1/3}}{n+1}$ convg.

d) $\sum_{n=2}^{\infty}\left[\ln\left(1+\frac{1}{n}\right)\right]^n$

Note $\ln(1+\frac{1}{n})>0$, Root Test.

$\lim_{n\to\infty}\left|[\ln(1+\frac{1}{n})]^n\right|^{1/n} = \lim_{n\to\infty}\ln(1+\frac{1}{n}) = \ln 1 = 0 < 1$

X absolutely convergent
___ conditionally convergent
___ divergent

5. A thinker . . . .

Fill in the blanks in the below statement of the Integral Test & Remainder Estimate:

Let $\sum a_n$ be a positive-term series. Find a function $f(x)$ such that

(1) $f(n) = a_n$ for each n

(2) f is decreasing for $x \geq 1$

(3) f is continuous for $x \geq 1$.

Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ either:

(1) both converge

(2) both diverge .

If they both converge, then $\sum_{n=1}^{\infty} a_n = S_N + R_N$ where

$$S_N \equiv \sum_{n=1}^{N} a_n \quad \text{and} \quad R_N \equiv \sum_{n=N+1}^{\infty} a_n .$$

Furthermore,

$$\boxed{\int_{N+1}^{\infty} f(x)\,dx} \leq R_N \leq \boxed{\int_{N}^{\infty} f(x)\,dx}$$

Now consider a sequence $\{b_n\}$ of positive numbers. We say that the *infinite product*

$$\prod_{n=1}^{\infty}(1 + b_n)$$

converges to P provided the associated *sequence* $\{P_N\}_{N=1}^{\infty}$ of *partial products* where

$$P_N \equiv \prod_{n=1}^{N}(1 + b_n)$$

converges to P. So "$\prod_{n=1}^{\infty}(1 + b_n)$ converges to P" is equivalent to

$$\lim_{N\to\infty} P_N = P$$

which is equivalent to

$$\lim_{N\to\infty} \ln(P_N) = \ln(P)$$

which is equivalent to

$$\lim_{N\to\infty} \sum_{n=1}^{N} \ln(1 + b_n) = \ln(P) .$$

Using (carefully) the Integral Test, determine whether or not the infinite series

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right)$$

converges. From the Integral Test Remainder Estimate, what can you conclude about the relation between $\underbrace{\prod_{n=1}^{N}\left(1 + \frac{1}{n^2}\right)}_{\equiv P_N}$ and $\underbrace{\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right)}_{\equiv P}$? Show your work on the next page.

- $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ converges to P $\iff$ $\lim_{N\to\infty} \sum_{n=1}^{N} \ln\left(1+\frac{1}{n^2}\right) = \ln P$.

- Integral Test. $f(x) = \ln\left(1+\frac{1}{x^2}\right)$. & $a_n = \ln\left(1+\frac{1}{n^2}\right) > 0$ NOTE

  1) $f(n) = a_n$ ok

  2) f is decreasing ok. as $x\uparrow$, $\frac{1}{x^2}\downarrow$ so $1+\frac{1}{x^2}\downarrow$ so $\ln\left(1+\frac{1}{x^2}\right)\downarrow$

  3) f is continuous ok.

- $\int \ln\left(1+\frac{1}{x^2}\right)dx \overset{\text{parts}}{=} x\ln\left(1+\frac{1}{x^2}\right) + 2\int \frac{dx}{x^2+1} = x\ln\left(1+\frac{1}{x^2}\right) + 2\tan^{-1}x + C$

$u = \ln\left(1+\frac{1}{x^2}\right)$	$dv = dx$
$du = \frac{x^2}{x^2+1}\cdot -2x^{-3}\,dx$	$v = x$

- $\lim_{N\to\infty} \int_1^N \ln\left(1+\frac{1}{x^2}\right)dx = \lim_{N\to\infty}\left[N\ln\left(1+\frac{1}{N^2}\right) + 2\tan^{-1}N - \ln 2 - 2\tan^{-1}1\right]$

  $= \left[\lim_{N\to\infty} \ln\left(1+\frac{1}{N^2}\right)^{N^2 \cdot \frac{1}{N}}\right] + \frac{\pi}{2} - \ln 2 - 2\cdot\frac{\pi}{4}$ problem(1a)

  $= \ln e^{1} - \ln 2 = 1/2 - \ln 2 < \infty$.

  (0,1)

- So $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ conv $\Rightarrow$ $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ $\boxed{\text{conv.}}$

- $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right) = \sum_{n=1}^{N} \ln\left(1+\frac{1}{n^2}\right) + \sum_{n=1}^{N} \ln\left(1+\frac{1}{n^2}\right)$

  $\equiv \qquad\qquad \|$

  $\ln P = \ln P_N + \sum_{n=1}^{N} \ln\left(1+\frac{1}{n^2}\right)$

$\Rightarrow \left|\ln P - \ln P_N\right| = \left|\sum_{n=1}^{N} \ln\left(1+\frac{1}{n^2}\right)\right| \leq \int_N^{\infty} \ln\left(1+\frac{1}{x^2}\right)dx$.