MATH 142 sections 004 & 005
FALL 1993 EXAM # 1

1. Find $\frac{dy}{dx}$ for:

1a) $y = \operatorname{arcsec} x$ Answer: $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

1b) $y = e^x$ Answer: $\frac{dy}{dx} = e^x$

1c) $y = x^e$ Answer: $\frac{dy}{dx} = ex^{e-1}$

1d) $y = 3^x$ Answer: $\frac{dy}{dx} = 3^x (\ln 3)$

$$y = 3^x$$
$$\ln y = \ln 3^x$$
$$\ln y = x \ln 3$$
$$D_x \ln y = D_x x \ln 3$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 3 \, D_x x$$
$$\frac{1}{y}\frac{dy}{dx} = \ln 3$$
$$\frac{dy}{dx} = y \ln 3$$

1e) $y = e^e$ Answer: $\frac{dy}{dx} = 0$

1f) $y = x^x$ Answer: $\frac{dy}{dx} = x^x \left[1 + \ln x \right]$

$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$D_x \ln y = D_x x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + 1 \cdot \ln x$$

1g) $y \ln x + xe^y = 3$ Answer: $\frac{dy}{dx} = \boxed{\dfrac{-e^y - \frac{y}{x}}{xe^y + \ln x}}$

$$D_x\left[y \ln x + x e^y \right] = D_x 3$$
$$y \cdot \frac{1}{x} + \frac{dy}{dx} \ln x + x e^y \frac{dy}{dx} + 1 \cdot e^y = 0$$
$$\frac{dy}{dx}\left[\ln x + x e^y \right] = -e^y - \frac{y}{x}$$

2. Evaluate the following 4 integrals. ⊛ hint: +C ...

Ex 1 pg 417. 2a) $\int \sin^3 x \cos^2 x \, dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

$$\int \sin^3 x \cos^2 x \, dx = \int (1-\cos^2 x)\cos^2 x (\sin x \, dx) = -\int (1-u^2) u^2 \, du$$
$$= \int (u^4 - u^2)\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned}}$$

Like an example from class!, 2b) $\int \ln(2x+7)\, dx = (x + \frac{7}{2}) \ln(2x+7) - x + C$

way # 1 $\int \ln(2x+7)\, dx = x \ln(2x+7) - \int \frac{2x \, dx}{2x+7} = x \ln(2x+7) - \left[\int \left(1 + \frac{-7}{2x+7}\right) dx \right]$ by long division

$u = \ln(2x+7)$ $dv = dx$

$du = \frac{1}{2x+7} \cdot 2dx$ $v = x$

$$= x \ln(2x+7) - x + \int \frac{7}{2x+7}\, dx = x \ln(2x+7) - x + \frac{7}{2} \ln(2x+7) + C$$

called like terms

way # 2 $\int \ln(2x+7)\, dx = \frac{1}{2}(2x+7)\ln(2x+7) - \int 1 \, dx = (x + \frac{7}{2})\ln(2x+7) - x + C$

$u = \ln(2x+7)$ $dv = dx$

$du = \frac{1}{2x+7} 2dx$ $v = \frac{1}{2}(2x+7)$

Ex 3 p. 438 2c) $\int \frac{dx}{(4x^2+9)^2} = \frac{1}{108}\left[\tan^{-1}\frac{2x}{3} + \frac{6x}{4x^2+9} \right] + C$

$$\int \frac{dx}{(4x^2+9)^2} = \int \frac{dx}{[(2x)^2 + 3^2]^2} = \int \frac{\frac{3}{2}\sec^2\theta \, d\theta}{[9\sec^2\theta]^2} = \frac{3}{2} \cdot \frac{1}{9} \cdot \frac{1}{9} \int \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = \frac{1}{54}\int \cos^2\theta \, d\theta$$

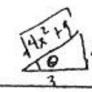
$2x = 3\tan\theta$

$2dx = 3\sec^2\theta$

$(2x)^2 + 3^2 = 9\tan^2\theta + 9$

$$= \frac{1}{54} \cdot \frac{1}{2} \int (1 + \cos 2\theta)\, d\theta = \frac{1}{108}\left[\theta + \frac{1}{2}\sin 2\theta \right] + C = \frac{1}{108}\left[\theta + \frac{1}{2} \cdot \cancel{2} \sin\theta \cos\theta \right]$$

$\frac{2x}{3} = \tan\theta = \frac{\sin\theta}{\cos\theta}$ $\Rightarrow$



$\Rightarrow \cos\theta = \frac{3}{\sqrt{4x^2+9}}$ & $\sin\theta = \frac{2x}{\sqrt{4x^2+9}}$

2d) $\int \frac{x^3 + 2x^2 + x + 1}{x^4 + x^2}\, dx = \ln|x| + -x^{-1} + \tan^{-1} x + C$

$$\frac{x^3 + 2x^2 + x + 1}{x^4 + x^2} = \frac{x^3 + 2x^2 + x + 1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2+1} = \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)(x^2)}{x^2(x^2+1)}$$

$\Rightarrow 1x^3 + 2x^2 + 1x + 1 = x^3(A + C) + x^2(B + D) + x(A) + 1(B) \Rightarrow$ $B = 1$, $A = 1$; $B + D = 2 \rightarrow D = 1$; $A + C = 1 \Rightarrow C = 0$

$$\int \frac{x^3 + 2x^2 + x + 1}{x^4 + x^2}\, dx = \int \left[\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^2+1} \right] dx$$

5. In 1921, President Warren G. Harding presentd Marie Curie a gift of 2 gram of radium on behalf of the women of the United States. Using the fact that the half-life of radium is 1656 years, determine how much of the original 2-gram gift is left today (in 1993). Your answer can involve exponentials and logs.

Answer: $2e^{-\frac{72\ln 2}{1656}}$ gram

units years & grams.

$A(t) = A_0 e^{kt}$

$\frac{1}{2}$ life info $\Rightarrow \frac{1}{2} A_0 = A_0 e^{k(1656)} \Rightarrow \frac{1}{2} = e^{k(1656)}$

$\Rightarrow 1656(k) = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$

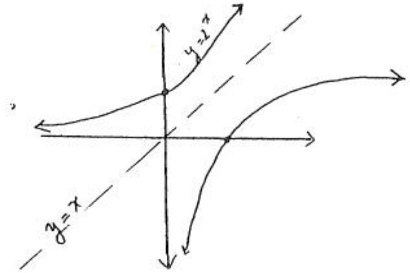
$\Rightarrow \boxed{k = -\frac{\ln 2}{1656}}$ so $A(t) = 2e^{-\frac{\ln 2}{1656} t}$

$1993 - 1921 = 72$

$A(72) = 2e^{-\frac{\ln 2}{1656}(72)}$

Marie Curie with President Warren G. Harding, 1921

3. Graph $y = 2^x$. The inverse of $y = 2^x$ is the function $\boxed{y = \log_2 x}$. Graph the inverse function of $y = 2^x$ on the same grid. Be sure to label your functions.



4.

4a) Let $y = f(x)$ and $y = g(x)$ be two functions defined for $x > 0$. If for all $x > 0$, you know that $f'(x) = g'(x)$, then what can you say about f and g?

they differ by a constant, i.e. there exists a constant C so that $f(x) = g(x) + C$

4b) Using part (4a) and the fact that $D_x \ln x = \frac{1}{x}$, show the following Law of Logarithm:

$$\boxed{\ln\left(\frac{1}{x}\right) = -\ln x}$$

CLEARLY explain your steps!

Let $f(x) = \ln\frac{1}{x}$ and $g(x) = -\ln x$.

Note that

$D_x f(x) = D_x \ln(x^{-1}) = \frac{1}{x^{-1}} \cdot (-1x^{-2}) = \frac{x^1}{1} \cdot \frac{-1}{x^2} = \frac{-1}{x}$

& $D_x g(x) = D_x(-\ln x) = -1\, D_x \ln x = -\frac{1}{x}$.

Since $f'(x) = g'(x)$, by (4a), there exists a constant C

so that $f(x) = g(x) + C$. So $\ln\left(\frac{1}{x}\right) = -\ln x + C$.

By letting $x = 1$, we get $\ln\left(\frac{1}{1}\right) = -\ln(1) + C$, i.e. $0 = -0 + C$

so $C = 0$. So $\ln\left(\frac{1}{x}\right) = -\ln x$.