MARK BOX		
#	Points	You
1	15	
2	15	
3	40	
4	15	
5	15	
6	15	
7	15	
8	10	
9	10	
10	15	
11	10	
12	10	
13	15	
Total	200	

MATH 142 FALL 1991 FINAL EXAM

NAME: Answer Key

SSN: ______

Instructions:

(1) To receive credit, you must work in a logical fashion, SHOW ALL YOUR WORK, INDICATE YOUR REASONING, and when applicable put your answer on the line (or in the box) provided.

(2) During this test, do not leave your seat. Raise your hand if you have a question. When you finish, turn your exam over, put your pencil down, and raise your hand.

(3) No "formula sheets" allowed. Calculators allowed.

(4) The "Mark Box" indicates the problems along with their points. Check that your copy of the exam has all of the problems.

1. Find $\frac{dy}{dx}$ for:

a) $y = e^x$ Answer: $\frac{dy}{dx} =$ e^{x}.

b) $y = x^x$ Answer: $\frac{dy}{dx} =$ $x^{x}(1 + \ln x)$

Logarithmic differentiation $\Rightarrow \ln y = \ln x^x = x \ln x \xrightarrow{D_x}$

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[x \ln x] = 1 \cdot \ln x + x \cdot \frac{1}{x} = 1 + \ln x$

c) $y \ln x + xe^y = 3$ Answer: $\frac{dy}{dx} =$ $-\left(\frac{y}{x} + e^y\right)\left(xe^y + \ln x\right)^{-1}$.

$\frac{d}{dx}\left(y \ln x + x e^y\right) = \frac{d}{dx}(3) \Rightarrow$

$\frac{dy}{dx} \ln x + y \frac{1}{x} + 1 e^y + x e^y \frac{dy}{dx} = 0$

$(\ln x + x e^y)\frac{dy}{dx} = -\frac{y}{x} - e^y$

2. ⊛ Find the limits of the below 3 functions. Indicate the reasoning behind your answer.

a) $\lim_{x \to 0+} \frac{\sin x}{x} = $ $\underline{1}$

$$\lim_{x \to 0^+} \frac{\sin x}{x} \xrightarrow[L'H]{\frac{0}{0}} \lim_{x \to 0^+} \frac{\cos x}{1} = 1$$

b) $\lim_{x \to 0+} x^x = $ $\underline{1}$

$$\lim_{x \to 0^+} \ln(x^x) = \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} -x = 0$$

$$x^x = e^{\ln(x^x)} \xrightarrow{x \to 0^+} e^0 = 1$$

c) $\lim_{x \to \infty} \sin 2\pi x = $ $\underline{DNE}$

oscillates between -1 & 1.

⊛ Find the limits of the below 3 sequences. Indicate your reasoning!

d) $\lim_{n \to \infty} \sin 2\pi n = $ $\underline{0}$

$\sin(2\pi n) = 0$ for all $n \in \mathbb{N}$

e) $\lim_{n \to \infty} \left(\frac{1}{2}\right)^n = $ $\underline{0}$

$\left|\frac{1}{2}\right| < 1$

f) $\lim_{n \to \infty} (-1)^n = $ $\underline{DNE}$

oscillates between -1 & 1

3. Evaluate the following 4 integrals. ⊛ hint: +C ...

a) $\int \frac{dx}{x(\ln x)^4} = \frac{1}{-3(\ln x)^3} + C$

(a)

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}}$$

$$\int \frac{dx}{x(\ln x)^4} = \int u^{-4}\, du = \frac{u^{-3}}{-3} + C$$

(b) $$\frac{3x+7}{x^2+4x+5} = \underbrace{\frac{\frac{3}{2}(2x+4)}{x^2+4x+5}}_{\substack{u = x^2+4x+5 \\ du = 2x+4}} + \underbrace{\frac{1}{(x+2)^2+1}}_{\boxed{\substack{v = x+2 \\ dv = dx}}}$$

$$\int \frac{3x+7}{x^2+4x+5}\, dx = \frac{3}{2}\int \frac{du}{u} + \int \frac{dv}{v^2+1}$$

$$= \frac{3}{2} \ln|u| + \arctan v + C$$

b) $\int \frac{3x+7}{x^2+4x+5}\, dx = \frac{3}{2} \ln|x^2+4x+5| + \arctan(x+2) + C$

⊛ show your work on the back of the previous page.

⊛ hint: split up into two integrals

$x^2 + 4x + 5 \Rightarrow b^2 - 4ac = 16^2 - 4(5) < 0 \Rightarrow$ irred. quad.

$x^2 + 4x + 5 = (x+2)^2 + 1$

c) $\int \frac{2\,dx}{(x-1)(x^2+1)} = \underline{\ln|x-1| - \frac{1}{2}\ln|x^2+1| - \arctan x} + C$

(c)

$$\frac{2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(x-1)(Bx+C)}{(x-1)(x^2+1)}$$

$$2 = A(x^2+1) + (x-1)(Bx+C) \xrightarrow{x=1} 2 = A(2) = A = 1$$

equate coeff:

x^2 : $0 = A + B \Rightarrow B = -1$

x : $0 = C - B \Rightarrow C = -1$

constant : $2 = A - C$

$$\int \frac{2\,dx}{(x-1)(x^2+1)} = \int \frac{dx}{x-1} + -1\,\frac{1}{2}\int \frac{2\,x\,dx}{x^2+1} + -1 \int \frac{dx}{x^2+1}$$

(d) $u = \sin x$ $\quad dv = e^x\,dx$ $\quad$ #1
$du = \cos x\,dx$ $\quad v = e^x$

$u = \cos x$ $\quad dv = e^x\,dx$ $\quad$ #2
$du = -\sin x\,dx$ $\quad v = e^x$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$
$$= e^x \sin x - [e^x \cos x - -\int e^x \sin x\,dx]$$
$$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

d) $\int e^x \sin x\,dx = \underline{\frac{1}{2}(e^x \sin x - e^x \cos x) + C}$

⊛ show your work on the <u>back</u> of the <u>previous</u> page.

4. Determine whether each of the following 2 series is absolutely convergent, conditionally convergent, or divergent. CLEARLY explain your reasoning and indicate the test(s) used. No credit will be given the work that does not make sense to us!!!!

a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

X absolutely convergent
___ conditionally convergent
___ divergent

Ratio Test (abs. conv.)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{e^n}{e^{n+1}} \right| = \frac{1}{e} \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \frac{1}{e} < 1.$$

b) $\sum_{n=1}^{\infty} \frac{n^3 - 7n^2 + 8}{12n^4 + 7}$

X absolutely convergent
___ conditionally convergent
___ divergent

LCT for abs conv $\leftarrow$ n big enough $\Rightarrow \frac{n^3 - 7n^2 + 8}{12n^4 + 7} > 0$.

and $\frac{n^3 - 7n^2 + 8}{12n^4 + 7} \overset{n \text{ big}}{\sim} \frac{n^3}{12n^4} = \frac{1}{2} \frac{1}{n} \Rightarrow b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3 - 7n^2 + 8}{12n^4 + 7} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^4 - 7n^3 + 8n}{12n^4 + 7}$$

$$= \frac{1}{12} \text{ and } 0 < \frac{1}{12} < \infty$$

5. Find the interval of convergence for each of the below 2 power series. Do not forget to 'check the endpoints'.

a) $\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n}$ has interval of convergence $[1, 2)$.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x-3)^{n+1}}{(2x-3)^n} \frac{n}{n+1}\right|$$

$$= |2x-3| \lim_{n\to\infty} \frac{n}{n+1} = 2\left|x-\frac{3}{2}\right| < 1$$

abs conv. when $\left|x-\frac{3}{2}\right| < \frac{1}{2} \Rightarrow$

endpts:

$x = 2$ $\quad \sum \frac{(2x-3)^n}{n} = \sum \frac{1}{n}$ divg, harmonic series

$x = 1$ $\quad \sum \frac{(2x-3)^n}{n} = \sum \frac{(-1)^n}{n}$ cond. conv., harm. series ⊕ AST

b) $\sum_{n=1}^{\infty} \frac{n!(x-4)^n}{n^2+7}$ has interval of convergence $\{4\}$.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{n!} \frac{(x-4)^{n+1}}{(x-4)^n} \frac{n^2+7}{(n+1)^2+7}\right|$$

$$= |x-4| \lim_{n\to\infty} \frac{(n+1)(n^2+7)}{(n+1)^2+7} = |x-4| \lim_{n\to\infty} \frac{n^3+?n^2+?n+?}{n^2+?n+7}$$

$$= |x-4| \cdot \infty$$

$\underline{x \neq 4}$ $\quad |x-4| \cdot \infty > 1$

$x = 4$ $\quad \sum_{n=1}^{\infty} \frac{n!\,(x-4)^n}{n^2+7} = \sum_{n=1}^{\infty} 0 = 0$

6. Find a power series expansion (namely the Taylor series) of the below functions about the point a. Be clever and use known Taylor Series. Write your answer in closed form. Indicate the values of x for which the expanion is valid. Show your work.

a) About $a = 0$: $xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$ valid for $x \in \mathbb{R}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

$$x e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x \cdot x^n}{n!}$$

b) About $a = 1$: $\frac{1}{5+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{6^{n+1}}$ valid for $x \in (-5, 7)$

$$\frac{1}{5+x} = \frac{1}{5+(x-1)+1} = \frac{1}{6+(x-1)} = \frac{1}{6} \frac{1}{1-\left(\frac{x-1}{-6}\right)}$$

$$\overset{(*)}{=} \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{x-1}{-6}\right)^n = \sum_{n=0}^{\infty} \frac{(x-1)^n}{6 \cdot (-1)^n (6)^n}$$

(*) valid when $\left|\frac{x-1}{-6}\right| < 1 \iff |x-1| < 6$

c) About $a = 0$: $\frac{(\cos x) - 1}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!}$ valid for $x \in \mathbb{R} / \{0\}$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos x - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{\cos x - 1}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!\, x}$$

7. Working through the steps below, find a power series representation for the given definite integral. For what values of x is this series valid? Express your answers in closed form.

a) $\dfrac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, $|t| < 1$

b) $\dfrac{1}{1+t^5} = \sum_{n=0}^{\infty} (t^5)^n = \sum_{n=0}^{\infty} t^{5n}$, $|t^5| < 1 \iff |t| < 1$

c) $\dfrac{t^3}{1+t^5} = \sum_{n=0}^{\infty} t^{5n+3}$, $|t| < 1$

d) $\displaystyle\int_0^x \frac{t^3}{1+t^5}\,dt = \sum_{n=0}^{\infty} \frac{x^{5n+4}}{5n+4}$ valid for $|x| < 1$.

⊛ Show your work below:

$$= \int_0^x \sum_{n=0}^{\infty} t^{5n+3}\,dt$$

$|t| < 1$

$$= \sum_{n=0}^{\infty} \int_0^x t^{5n+3}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{t^{5n+4}}{5n+4}\Bigg|_{t=0}^{t=x}$$

$$= \left[\sum_{n=0}^{\infty} \frac{x^{5n+4}}{5n+4}\right] - \Bigg[\underbrace{\sum_{n=0}^{\infty} \frac{0^{5n+4}}{5n+4}}_{=0}\Bigg]$$

8. Fun with e^{-1}

a) Express the number e^{-1} as an infinite sum. ANSWER: $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

b) To 2 decimal places, approximate e^{-1} with a partial sum of the above infinite sum, using the fewest possible number of terms for which an appropriate estimate test (namely the alternating Series test - error estimate. test) guarantees the desired accuracy. Leave your answer as a sum

⊛ Answer $e^{-1} \approx \sum_{n=0}^{5} \frac{(-1)^n}{n!}$

$$\left| \sum_{n=0}^{\infty} (-1)^n a_n - \sum_{n=0}^{N} (-1)^n a_n \right| \leq a_{N+1} \quad \text{for } a_n > 0.$$

$$a_{N+1} = \frac{1}{(N+1)!} \underset{\substack{\uparrow \\ \text{want}}}{\leq} 0.005 = \frac{5}{1000} = \frac{1}{200}$$

2 decimal places

$$\Updownarrow$$

$$200 \leq (N+1)!$$

$4! = 24$

$5! = 120$

$\boxed{6! = 720}$ $\rightarrow N+1 = 6 \Rightarrow N = 5$

9. Fun with Mr. Taylor

a) Find the 5th ~~degree~~ order Taylor polynomial $P_5(x)$ and the remainder term $R_5(x)$ about the point $a = 1$ for the function $f(x) = \ln x$.

Answer: $P_5(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$

$R_5(x) = -\frac{(x-1)^6}{6z^6}$ where z is between 1 and x

(a)

$f(x) = \ln x$ $\qquad f(1) = 0$

$f^{(1)}(x) = x^{-1}$ $\qquad f^{(1)}(1) = 1$

$f^{(2)}(x) = -x^{-2}$ $\qquad f^{(2)}(1) = -1$

$f^{(3)}(x) = 2x^{-3}$ $\qquad f^{(3)}(1) = 2$

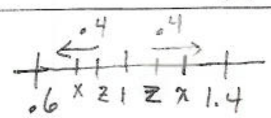
$f^{(4)}(x) = -2\cdot 3\, x^{-4}$ $\qquad f^{(4)}(1) = -3!$

$f^{(5)}(x) = 2\cdot 3\cdot 4\, x^{-5}$ $\qquad f^{(5)}(1) = 4!$

$$P_5(x) = 0 + 1(x-1)^1 + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5$$

$$R_5(x) = \frac{f^{(6)}(z)}{6!}(x-1)^6 = \frac{-5!\,z^{-6}}{6!}(x-1)^6 = -\frac{(x-1)^6}{6z^6}$$

(b)



$|x-1| \le .4$ and $.6 \le |z| \le 1.4$

$$|R_5(x)| = \frac{1}{6}\frac{|x-1|^6}{|z|^6} \le \frac{1}{6}\frac{(.4)^6}{(.6)^6} = \frac{1}{6}\left(\frac{2}{3}\right)^6 \le .015 < .05$$

1

b) In part a), to how many decimal places of accuracy does Taylor's formula guarantee that $P_5(x)$ approximate $f(x) = \ln x$ for x between .6 and 1.4? Show your work on the back of previous page.

Answer: 1 decimal places of accuracy

10. Consider the parametric curve described by $x = 2t^2 + 1$ and $y = 3t^3 + 2$.

a) The equation of the line tangent to this curve at the point $t = 1$ is

$(y-5) = \frac{9}{4}(x-3)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2}{4t} = \frac{9}{4}t \stackrel{t=1}{=} \frac{9}{4}$$

$$y(1) = 3(1)^3 + 2 = 5$$

$$x(1) = 2(1)^2 + 1 = 3$$

b) Find the second derivative of this curve at the point $t = 1$.

⊛ ANSWER: $\frac{d^2y}{dx^2} = \frac{9}{16}$.

Is the curve concave upward or downward at $t = 1$? cc up .

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

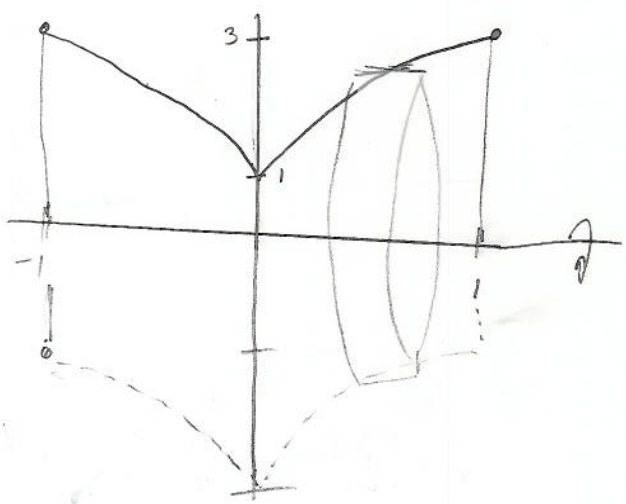
$$= \frac{9/4}{4t} = \frac{9}{4} \cdot \frac{1}{4t} = \frac{9}{16t}$$

11. Let V be the volume obtained by revolving about the x-axis the region that lies between the x-axis and the parametric curve given by $x = t^3$ and $y = 2t^2 + 1$ for $-1 \le t \le 1$. Express V as an integral (but do NOT perform the integration!)

⊛ Draw a helpful rough sketch showing a typical element.

⊛ Answer: $V = \int_{t=-1}^{t=1} 3\pi t^2 (2t^2+1)^2 \, dt$

$y = 2t^2 + 1$ and $x^{1/3} = t \Rightarrow y = 2x^{2/3} + 1$



t	(x,y)
-1	$(-1\ 3)$
1	$(1,3)$

$V_{\text{typical element}} = \pi (\text{radius})^2 (\text{height}) = \pi y^2 \Delta x$

$$V_x = \int_{x=a}^{x=b} \pi y^2 \, dx$$

$$V = \int_{t=-1}^{t=1} \pi (2t^2+1)^2 \, (3t^2 \, dt)$$

12. Consider the vector-valued functions (for $-\infty < t < \infty$):

$$\mathbf{u}(t) = 8t\mathbf{i} + 1\mathbf{j} \quad \text{and} \quad \mathbf{v}(t) = 2t\mathbf{i} + 1\mathbf{j}$$

a) How long is the vector $\mathbf{u}(t)$ when $t = 1$? ANS: $|\mathbf{u}(1)| =$ $\sqrt{65}$.

$|\vec{u}(1)| = |\langle 8, 1 \rangle| = \sqrt{8^2 + 1^2}$

b) Find the dot product $\mathbf{u}(t) \cdot \mathbf{v}(t)$. ANS: $\mathbf{u}(t) \cdot \mathbf{v}(t) =$ $16t^2 + 1$.

$\langle 8t, 1 \rangle \cdot \langle 2t, 1 \rangle = 16t^2 + 1$

c) We know that two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} =$ 0.

d) For what values of t are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ perpendicular? ANS: no such t.

$\vec{u}(t) \cdot \vec{v}(t) = 16t^2 + 1 = 0$

$t^2 = -\frac{1}{16}$

13. A puffo (that's Italian for smurf) moves along with acceleration described by $\mathbf{a}(t) = t\mathbf{i} + t^2\mathbf{j}$. His initial velocity is $\mathbf{v}_0 = \mathbf{0}$ and initial position of $\mathbf{r}_0 = \mathbf{i}$. Find his velocity vector $\mathbf{v}(t)$ and position vector $\mathbf{r}(t)$.

⊛ $\mathbf{v}(t) = \langle \frac{t^2}{2}, \frac{t^3}{3} \rangle$.

⊛ $\mathbf{r}(t) = \langle 1 + \frac{t^3}{6}, \frac{t^4}{12} \rangle$.

$$\vec{a}(t) = \langle t, t^2 \rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \frac{t^2}{2}, \frac{t^3}{3} \rangle + \underbrace{\langle a, b \rangle}_{\text{constant vector}}$$

$$\langle 0, 0 \rangle = \vec{v}(0) = \langle \frac{0^2}{2}, \frac{0^2}{3} \rangle + \langle a, b \rangle = \langle a, b \rangle$$

$$\vec{v}(t) = \langle \frac{t^2}{2}, \frac{t^3}{3} \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{t^3}{6}, \frac{t^4}{12} \rangle + \underbrace{\langle c, d \rangle}_{\text{Constant vector}}$$

$$\langle 1, 0 \rangle = \vec{r}(0) = \langle \frac{0^3}{6}, \frac{0^4}{12} \rangle + \langle c, d \rangle = \langle c, d \rangle$$

$$\vec{r}(t) = \langle \frac{t^3}{6}, \frac{t^4}{12} \rangle + \langle 1, 0 \rangle$$