MATH 142 Answer to Exam 3

1. Power Series Expansion about $a=0$.

(a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. (b) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all x

(d) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$ (c) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x

2. More Power Series

(*) $\cos(x-\frac{\pi}{2}) = \cos x \cos(-\frac{\pi}{2}) - \sin x \sin(-\frac{\pi}{2})$
$= 0 - -\sin x = \sin x$

$a=0$ (a) $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$ for all x

$a=\frac{\pi}{2}$ (c) $\sin x \overset{(*)}{=} \cos(x-\frac{\pi}{2}) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$ good for all x.

$a=1$ (b) $\frac{1}{5+x} = \frac{1}{6+(x-1)} = \frac{1}{6} \frac{1}{1-(\frac{x-1}{6})} = \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{x-1}{6}\right)^n = \sum_{n=0}^{\infty} \frac{(x-1)^n}{6^{n+1}}$.

good for $\left|\frac{x-1}{6}\right| < 1$ so $|x-1| < 6$.

3. Series

(a) $\sum \frac{(-1)^n}{n}$ cond. conv. since
① $\sum \frac{(-1)^n}{n}$ conv by alt. series test since $\frac{1}{n} \downarrow 0$
② $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ divg. – harmonic series

(b) $\sum \frac{n^2+8}{10n^3+7}$ abs. conv Note that $\left|\frac{n^2+8}{10n^3+7}\right| = \frac{n^2+8}{10n^3+7}$

Limit Comparison Test : $\frac{n^2+8}{10n^3+7} \sim \frac{n^2}{10n^3} = \frac{1}{10n^1}$ so $b_n = \frac{1}{10n}$ (or $b_n = \frac{1}{n}$)

$\lim_{n\to\infty} \frac{n^2+8}{10n^3+7} \cdot \frac{10n}{1} = \lim_{n\to\infty} \frac{10n^3+80n}{10n^3+7} = \lim_{n\to\infty} \frac{10+\frac{80}{n}}{10+\frac{7}{n^3}} = 1 \neq 0, \infty$

So $\sum \frac{n^2+8}{10n^3+7}$ does the same $\sum \frac{1}{10n}$, which conv. (harmonic Series)

(c) $\sum \frac{\ln n}{n^2}$ abs conv. Note $\left|\frac{\ln n}{n^2}\right| = \frac{\ln n}{n^2}$

Comparison Test : $\frac{\ln n}{n^2} \overset{n \text{ big}}{\leq} \frac{n^q}{n^2} \underset{q=1/2}{=} \frac{1}{n^{3/2}}$ others work and $\sum \frac{1}{n^{3/2}}$ conv.

so $\sum \frac{\ln n}{n^2} \leq \sum \left(\frac{1}{n}\right)^{3/2}$ since p-series, $p = 3/2 >$ So $\sum \frac{\ln n}{n^2}$ conv.

(d) $\sum \frac{(-1)^n}{n!}$ abs conv

ratio test $\rho = \lim_{n\to\infty} \left| \frac{n!}{\dots} \right| = \lim \frac{1}{\dots} = 0 < 1$.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \approx \sum_{n=1}^{N} \frac{(-1)^n}{n}$ within $.0001 = \frac{1}{10,000}$ Find N.

Use alternating Series Error Estimate test.
Here $a_n = \frac{1}{n} \downarrow 0$ so we can use the test.

Know $|R_N| < a_{N+1} = \frac{1}{N+1} \overset{\text{want}}{\leq} .0001 = \frac{1}{10,000}$

so $10,000 \leq N+1$

so $9,999 \leq N$

so can choose $\boxed{N = 9,999.}$

Remark: If you put $\frac{1}{N+1} < .0001$ then you get $10,000 < N+1$ so $9,999 < N$ so choose $\boxed{N = 10,000}$
We will also accept this answer

(5a) We know: if $\lim_{n\to\infty} a_n \neq 0$ then Σa_n divg.
But if $\lim_{n\to\infty} a_n = 0$, then we know nothing about Σa_n

eg $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\Sigma \frac{1}{n} = \infty$

and $\lim_{n\to\infty} \frac{1}{n^2} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$

It depends on how *fast* a_n goes to zero.

(5b) Many answers. Along one line of thought:

$a_n = \frac{(-1)^n}{\sqrt{n}}$ $\quad b_n = \frac{(-1)^n}{\sqrt{n}}$ $\quad$ so $\quad a_n b_n = \frac{(-1)^n}{\sqrt{n}} \frac{(-1)^n}{\sqrt{n}} = \frac{+1}{n}$

Then Σa_n and Σb_n conv. by the Alt. Series Test

But $\Sigma a_n b_n = \Sigma \frac{1}{n}$ divg ... harmonic Series.